Bessel Functions

http://en.wikipedia.org/wiki/Bessel_function

Spherical Bessel Functions

Recursion Relation

$\frac{2n+1}{x} j_n(x) = j_{n-1}(x) + j_{n+1}(x)$

First Kind

$$
\begin{aligned}
j_0(x) &= \frac{\sin x}{x} \\
j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} \\
j_2(x) &= \left(\frac{3}{x^2} - 1\right)\frac{\sin x}{x} - \frac{3\cos x}{x^2} \\
j_3(x) &= \left(\frac{15}{x^3} - \frac{6}{x}\right)\frac{\sin x}{x} - \left(\frac{15}{x^2} - 1\right)\frac{\cos x}{x} \\
j_4(x) &= \left(\frac{105}{x^4} - \frac{45}{x^2} + 1\right)\frac{\sin x}{x} + \left(-\frac{105}{x^2} + 10\right)\frac{\cos x}{x^2}
\end{aligned}
$$

$$\boxed{j_2(x) = \frac{3}{x}\left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right) - \frac{\sin x}{x} = \frac{\sin x}{x}\left(\frac{3}{x^2} - 1\right) - \frac{\cos x}{x^2} 3}$$

$$j_3(x) = \frac{5}{x}\left(\left(\frac{3}{x^2} - 1\right)\frac{\sin x}{x} - \frac{3\cos x}{x^2}\right) - \left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right) =$$

$$\frac{1}{x}\cos x - \frac{1}{x^2}\sin x + \frac{5}{x}\left(\frac{1}{x}(\sin x)\left(\frac{3}{x^2} - 1\right) - \frac{3}{x^2}\cos x\right)$$

$$\boxed{j_3(x) = \frac{1}{x^4}(15\sin x - 15x\cos x + x^3\cos x - 6x^2\sin x) = \left(\frac{15}{x^3} - \frac{6}{x}\right)\frac{\sin x}{x} + \left(-\frac{15}{x^2} + 1\right)\frac{\cos x}{x}}$$

$$j_4(x) = \frac{7}{x}\left(\left(\frac{15}{x^3} - \frac{6}{x}\right)\frac{\sin x}{x} - \left(\frac{15}{x^2} - 1\right)\frac{\cos x}{x}\right) - \left(\left(\frac{3}{x^2} - 1\right)\frac{\sin x}{x} - \frac{3\cos x}{x^2}\right) =$$

$$\frac{3}{x^2}\cos x - \frac{1}{x}(\sin x)\left(\frac{3}{x^2} - 1\right) + \frac{7}{x}\left(\frac{1}{x}(\sin x)\left(\frac{15}{x^3} - \frac{6}{x}\right) - \frac{1}{x}(\cos x)\left(\frac{15}{x^2} - 1\right)\right)$$

$$\boxed{j_4(x) = \frac{1}{x^5}(105\sin x - 105x\cos x + 10x^3\cos x - 45x^2\sin x + x^4\sin x)}$$

$$\boxed{j_4(x) = \left(\frac{105}{x^4} - \frac{45}{x^2} + 1\right)\frac{\sin x}{x} + \left(-\frac{105}{x^2} + 10\right)\frac{\cos x}{x^2}}$$

Second Kind

$$
\begin{aligned}
n_0(x) &= -\frac{\cos x}{x} \\
n_1(x) &= -\frac{\cos x}{x^2} - \frac{\sin x}{x} \\
n_2(x) &= \left(-\frac{3}{x^2} + 1\right)\frac{\cos x}{x} - \frac{3\sin x}{x^2} \\
n_3(x) &= \left(-\frac{15}{x^3} + \frac{6}{x}\right)\frac{\cos x}{x} + \frac{\sin x}{x}\left(-\frac{15}{x^2} + 1\right) \\
n_4(x) &= \left(-\frac{105}{x^2} + 10\right)\frac{\sin x}{x^2} + \left(-\frac{105}{x^4} + \frac{45}{x^2} - 1\right)\frac{\cos x}{x}
\end{aligned}
$$

$$\boxed{n_2(x) = \frac{3}{x}\left(-\frac{\cos x}{x^2} - \frac{\sin x}{x}\right) - \left(-\frac{\cos x}{x}\right) = \left(-\frac{3}{x^2} + 1\right)\frac{\cos x}{x} - \frac{3\sin x}{x^2}}$$

$$n_3(x) = \frac{5}{x}\left(\left(\left(-\frac{3}{x^2}+1\right)\frac{\cos x}{x} - \frac{3\sin x}{x^2}\right) - \left(-\frac{\cos x}{x^2} - \frac{\sin x}{x}\right)\right) =$$

$$\frac{1}{x^2}\cos x + \frac{1}{x}\sin x + \frac{5}{x}\left(\frac{1}{x}(\cos x)\left(1-\frac{3}{x^2}\right) - \frac{3}{x^2}\sin x\right)$$

$$\boxed{n_3(x) = \frac{1}{x^4}(6x^2\cos x - 15x\sin x - 15\cos x + x^3\sin x) = \left(-\frac{15}{x^3}+\frac{6}{x}\right)\frac{\cos x}{x} + \frac{\sin x}{x}\left(-\frac{15}{x^2}+1\right)}$$

$$n_4(x) = \frac{7}{x}\left(\left(\left(-\frac{15}{x^3}+\frac{6}{x}\right)\frac{\cos x}{x} + \frac{\sin x}{x}\left(-\frac{15}{x^2}+1\right)\right) - \left(\left(-\frac{3}{x^2}+1\right)\frac{\cos x}{x} - \frac{3\sin x}{x^2}\right)\right) =$$

$$\frac{3}{x^2}\sin x - \frac{1}{x}(\cos x)\left(1-\frac{3}{x^2}\right) + \frac{7}{x}\left(\frac{1}{x}(\sin x)\left(1-\frac{15}{x^2}\right) + \frac{1}{x}(\cos x)\left(\frac{6}{x}-\frac{15}{x^3}\right)\right) =$$

$$\boxed{n_4(x) = \frac{1}{x^5}(45x^2\cos x - 105x\sin x - 105\cos x - x^4\cos x + 10x^3\sin x)}$$

$$\boxed{n_4(x) = \left(-\frac{105}{x^2}+10\right)\frac{\sin x}{x^2} + \left(-\frac{105}{x^4}+\frac{45}{x^2}-1\right)\frac{\cos x}{x}}$$

Barrier Penetration Factor

$$\boxed{V_\ell(r_0k) = \frac{1}{(r_0k)^2\left[j_\ell^2(r_0k)+n_\ell^2(r_0k)\right]}}$$

$$j_0^2 + n_0^2 = \left(\frac{\sin x}{x}\right)^2 + \left(-\frac{\cos x}{x}\right)^2 = j_0^2 + n_0^2 = \boxed{\frac{1}{x^2}}$$

$$j_1^2 + n_1^2 = \left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right)^2 + \left(-\frac{\cos x}{x^2} - \frac{\sin x}{x}\right)^2 = \boxed{\frac{1}{x^4}(x^2+1)}$$

$$j_2^2 + n_2^2 = \left(\left(\frac{3}{x^2}-1\right)\frac{\sin x}{x} - \frac{3\cos x}{x^2}\right)^2 + \left(\left(-\frac{3}{x^2}+1\right)\frac{\cos x}{x} - \frac{3\sin x}{x^2}\right)^2 = \boxed{\frac{1}{x^6}(3x^2+x^4+9)}$$

$$j_3^2 + n_3^2 = \left(\left(\frac{15}{x^3}-\frac{6}{x}\right)\frac{\sin x}{x} - \left(\frac{15}{x^2}-1\right)\frac{\cos x}{x}\right)^2 + \left(\left(-\frac{15}{x^3}+\frac{6}{x}\right)\frac{\cos x}{x} + \frac{\sin x}{x}\left(-\frac{15}{x^2}+1\right)\right)^2 =$$

$$.j_3^2 + n_3^2 = \boxed{\frac{1}{x^8}(45x^2+6x^4+x^6+225)}$$

$$j_4^2 + n_4^2 = \boxed{\frac{1}{x^{10}}(1575x^2+135x^4+10x^6+x^8+11\,025)}$$