Realistic Functions for Nonlinear Systems

L. David Roper, roperld@vt.edu
http://arts.bev.net/RoperLDavid/

Contents

The following functions are useful when doing empirical studies of nonlinear physical and sociological systems. In many cases empirical data can be fitted to one or a combination of the functions given here. In some cases such a fit can be used to reliably extrapolate outside the range of the data.

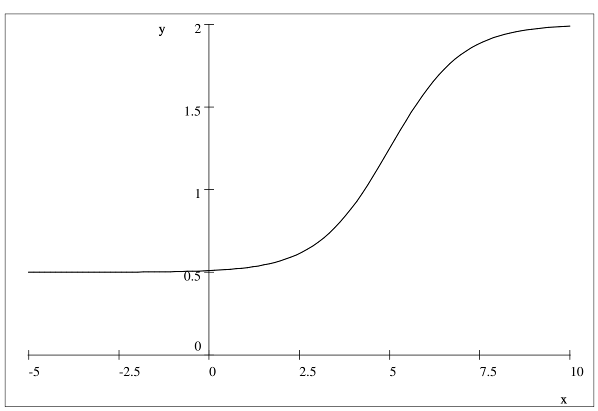
Asymptotic Functions

Nonlinear systems, which comprise most physical and sociological systems, often have behaviors that are asymptotic; i.e., they move from one steady state to another steady state.

Hyperbolic-Tangent Function

$$\boxed{H(x) = \frac{1}{2}\left[a + b + (b-a)\tanh\left(\frac{x-c}{d}\right)\right]}$$

Plot $\frac{1}{2}\left[(.5+2)+(2-.5)\tanh\left(\frac{x-5}{2}\right)\right]$:



Several changes between N different levels can be represented by N-1 hyperbolic-tangent functions:

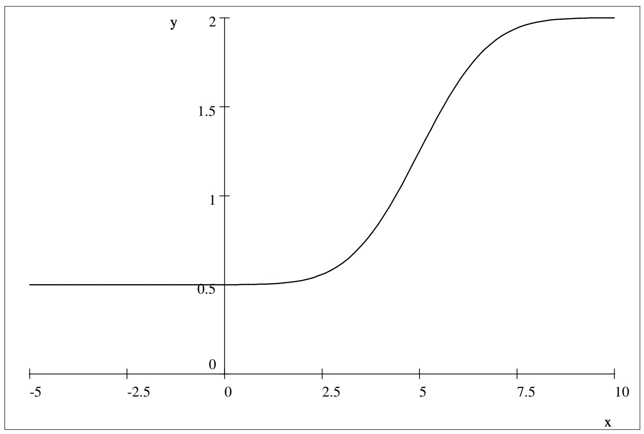
$$H_N(x) = \frac{1}{2}\left[a_1 + a_N + \sum_{n=1}^{N-1}(a_{n+1} - a_n)\tanh\left(\frac{x-c_n}{d_n}\right)\right]$$

http://en.wikipedia.org/wiki/Hyperbolic_tangent
http://www.roperld.com/science/Mathematics/HyperbolicTangentWorld.htm

Error Function

$$E(x) = \frac{1}{2}\left[a + b + (b-a)\,\text{erf}\left(\frac{x-a}{b}\right)\right]$$ where $\text{erf}(x) = \frac{1}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$

Plot $\frac{1}{2}\left[(.5+2) + (2-.5)\,\text{erf}\left(\frac{x-5}{2}\right)\right]$:



The error function is the integral of the Gaussian function, which is discussed below.
Several changes between N different levels can be represented by N-1 error functions:

$$E_N(x) = \frac{1}{2}\left[a_1 + a_N + \sum_{n=1}^{N-1}(a_{n+1} - a_n)\,\text{erf}\left(\frac{x-c_n}{d_n}\right)\right]$$

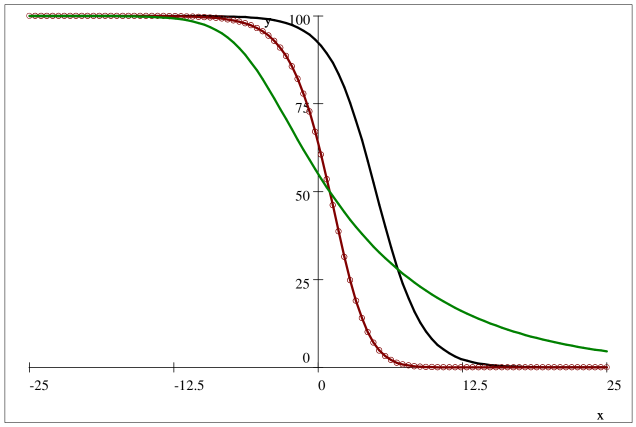
http://en.wikipedia.org/wiki/Error_function

Verhulst Function

$$v(x) = \frac{Q_\infty}{[1+(2^n-1)\exp x]^{\frac{1}{n}}}$$

Symmetric: $n = 1$, Left skew: $n < 1$, Right skew: $n > 1$

Plot $\frac{100}{\left[1+(2^1-1)\exp\left(\frac{t-5}{2}\right)\right]^{\frac{1}{1}}}, \frac{100}{\left[1+(2^{0.5}-1)\exp\left(\frac{t-1}{2}\right)\right]^{\frac{1}{0.5}}}, \frac{100}{\left[1+(2^5-1)\exp\left(\frac{t-1}{2}\right)\right]^{\frac{1}{5}}}$:



Solid = $(n = 1)$, dots = $(n = 0.5)$, dash = $(n = 5)$.

For a level-changing function:

$$\boxed{Q(x) = \frac{1}{2}\left[a + b + (a-b)v\left(\frac{x-c}{d}\right)\right]}.$$

For changes between N levels:

$$\boxed{Q_N(x) = \frac{1}{2}\left[a_1 + a_N + \sum_{n=1}^{N-1}(a_n - a_{n+1})v\left(\frac{x-c_n}{d_n}\right)\right]}.$$

http://www.roperld.com/science/minerals/VerhulstFunction.htm

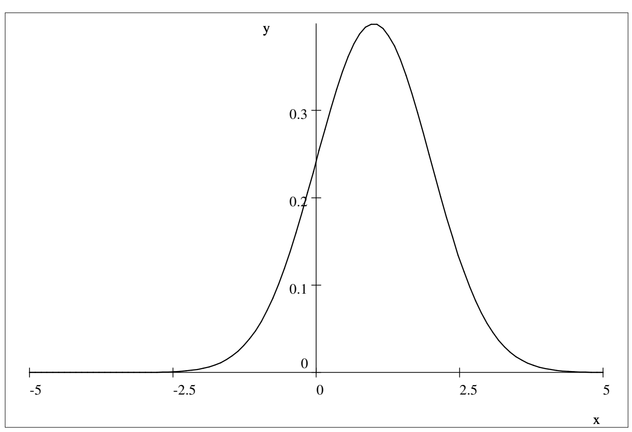
Peaked Functions

Nonlinear systems, which comprise most physical and sociological systems, often have behaviors that are peaked. The peaks are the derivatives of moves from one steady state to another steady state.

Gaussian Function

$$\boxed{G(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{(x-b)^2}{2\sigma^2}\right]}$$

Plot $\frac{1}{1\sqrt{2\pi}} \exp\left[-\frac{(x-1)^2}{2(1^2)}\right]$:



The Gaussian function is proportional to the derivative of the error function:

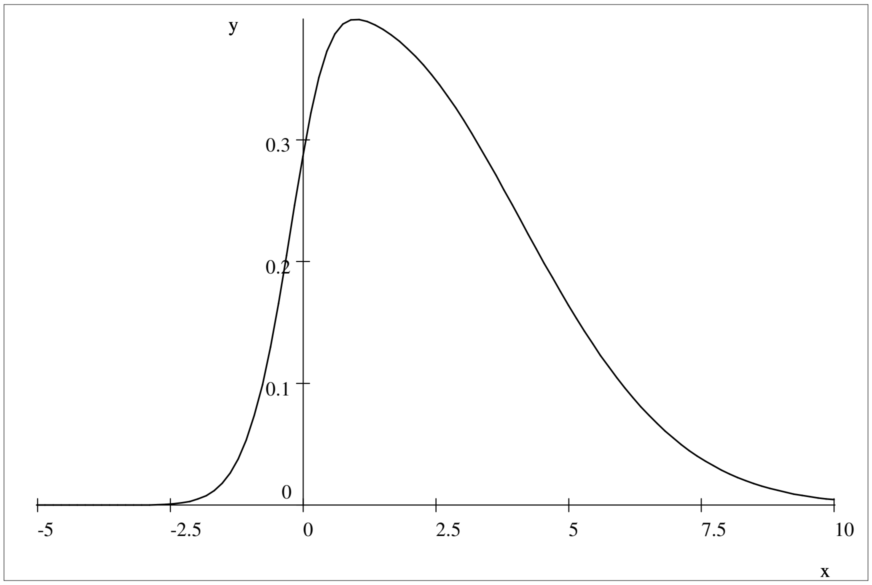
$$\frac{d}{dx}\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}$$

http://en.wikipedia.org/wiki/Gaussian_function

Asymmetric Gaussian Function

$$G_a(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{(x-b)^2}{2\left(\left[(\sigma+d)+(d-\sigma)\tanh\left(\frac{x-b}{c}\right)\right]/2\right)^2} \right]$$

Plot $\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(x-1)^2}{2\left(\left[(1+3)+(3-1)\tanh\left(\frac{x-1}{1}\right)\right]/2\right)^2} \right]$:



This uses the hyperbolic-tangent function to change the Gaussian width from σ to d.

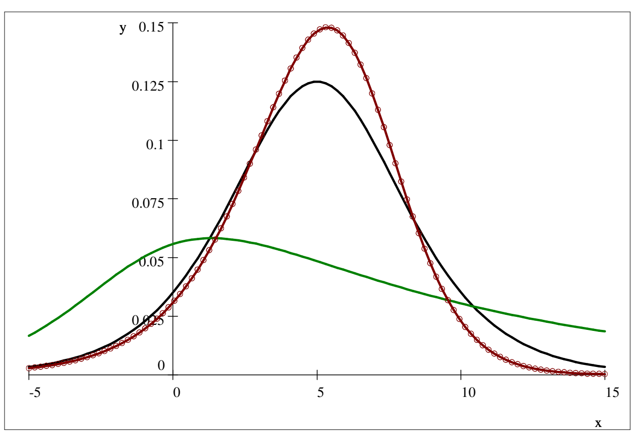
Verhulst Function

$$V(x) = \frac{Q_\infty}{n\tau} \frac{(2^n-1)\exp\left(\frac{t-a}{b}\right)}{\left[1+(2^n-1)\exp\left(\frac{t-a}{b}\right)\right]^{\frac{n+1}{n}}}$$

Symmetric: $n = 1$, Left skew: $n < 1$, Right skew: $n > 1$

Plot

$\frac{1}{1 \cdot 2} \frac{\left(2^1-1\right)\exp\left(\frac{t-5}{2}\right)}{\left[1+\left(2^1-1\right)\exp\left(\frac{t-5}{2}\right)\right]^{\frac{1+1}{1}}}, \frac{1}{0.5 \cdot 2} \frac{\left(2^{0.5}-1\right)\exp\left(\frac{t-5}{2}\right)}{\left[1+\left(2^{0.5}-1\right)\exp\left(\frac{t-5}{2}\right)\right]^{\frac{0.5+1}{0.5}}}, \frac{1}{5 \cdot 2} \frac{(2^5-1)\exp\left(\frac{t-5}{2}\right)}{\left[1+(2^5-1)\exp\left(\frac{t-5}{2}\right)\right]^{\frac{5+1}{5}}}$:



Solid = $(n = 1)$, dots = $(n = 0.5)$, dash = $(n = 5)$.

The peak occurs at $\boxed{t_m = a + b\ln\left(\frac{n}{2^n - 1}\right)}$

and the peak value is $\boxed{V_m = \frac{Q_\infty}{b(n+1)^{\frac{n+1}{n}}}}$.

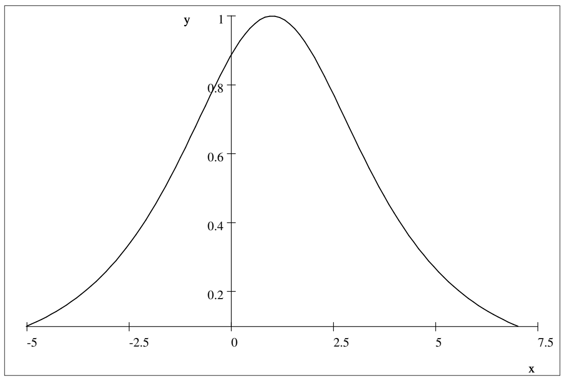
Note that $V(x) = \frac{dQ(x)}{dx}$.

http://www.roperld.com/science/minerals/VerhulstFunction.htm

Hyperbolic-Secant Function

$C(x) = \operatorname{sech}\left(\frac{x-a}{b}\right)$

Plot $\operatorname{sech}\left(\frac{x-1}{2}\right)$:



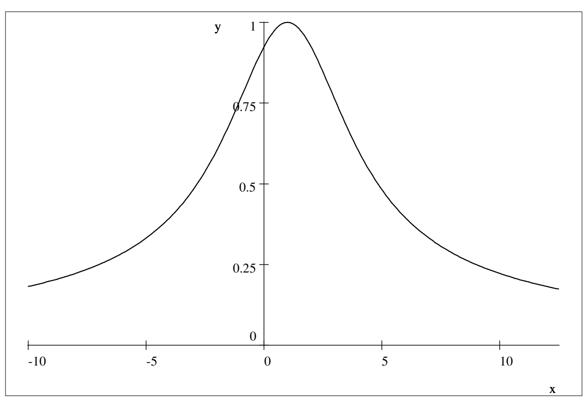
Note that $\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$.

http://en.wikipedia.org/wiki/Hyperbolic_tangent

Hyperbolic Tangent Over x

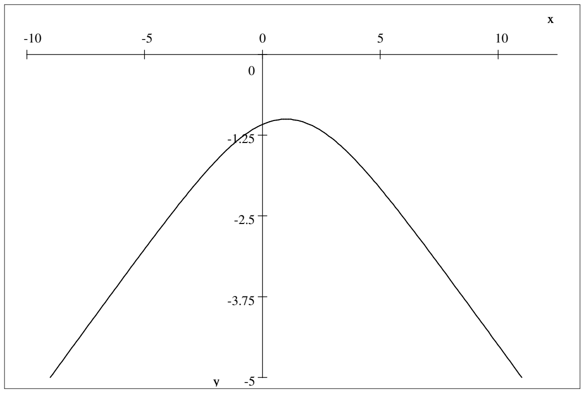
$\boxed{T(x) = \frac{\tanh\left(\frac{x-a}{b}\right)}{\frac{x-a}{b}}}$

Plot $\frac{\tanh\left(\frac{x-1}{2}\right)}{\frac{x-1}{2}}$:



Another interesting peaked function can be obtained by inverting this function with a minus sign: $\boxed{\frac{-\frac{x-a}{b}}{\tanh\left(\frac{x-a}{b}\right)}}$.

Plot $\frac{-\frac{x-1}{2}}{\tanh\left(\frac{x-1}{2}\right)}$

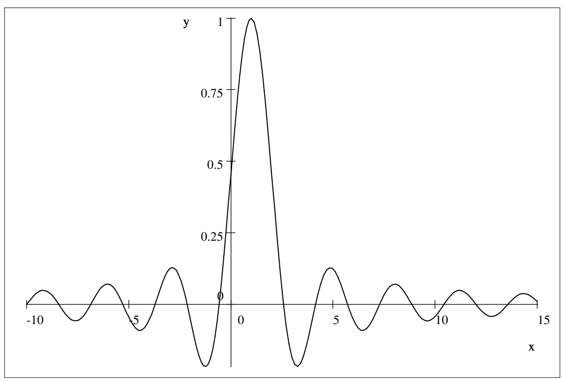


Transitory Oscillations

A simple transitory oscillation is

$$\boxed{C\frac{\sin(b(x-a))}{b(x-a)}}$$

Plot $\frac{\sin(2(x-1))}{2(x-1)}$:



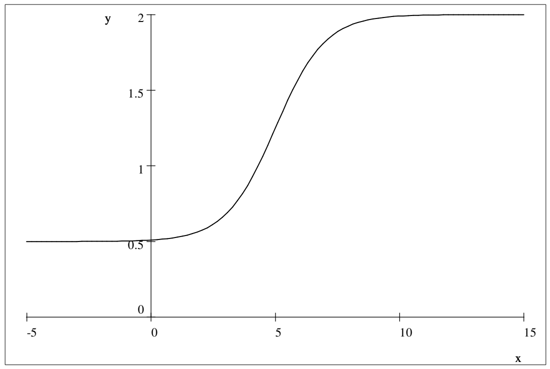
Oscillations Near Asymptotes

Often as a nonlinear system approaches an asymptote oscillations occur. A possible fit function might be

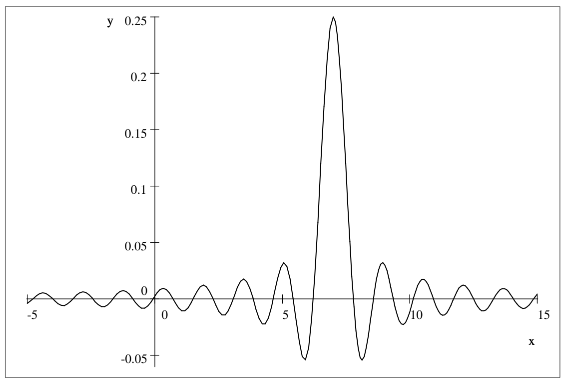
$$\boxed{\frac{1}{2}\left[a+b+(b-a)\tanh\left(\frac{x-c}{d}\right)\right]+\frac{0.25}{2d(x-c-d)}\sin[2d(x-c-d)]}$$

A numerical example is

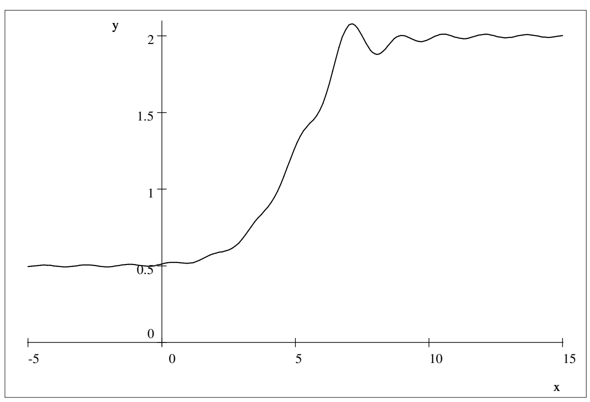
$\frac{1}{2}\left((.5+2)+(2-.5)\tanh\left(\frac{x-5}{2}\right)\right)$:



and $\frac{0.25}{4(x-7)}\sin(4(x-7))$:



Combined: $\frac{1}{2}\left((.5+2)+(2-.5)\tanh\left(\frac{x-5}{2}\right)\right)+\frac{.25}{4(x-7)}\sin(4(x-7))$:



A general equation might be

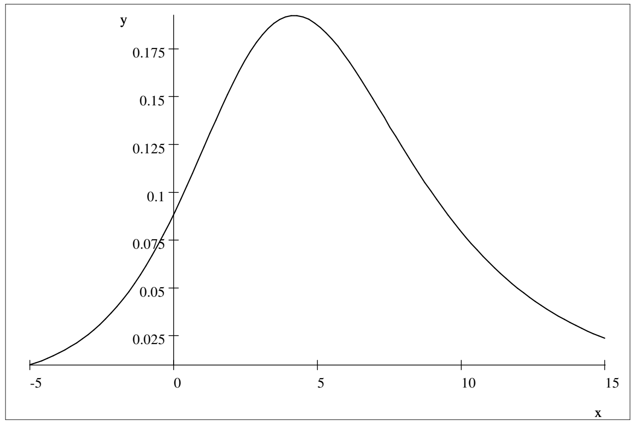
$$\boxed{\frac{1}{2}\left[a+b+(b-a)\tanh\left(\frac{x-c}{d}\right)\right]+\frac{g}{2e(x-f)}\sin[2e(x-f)]}$$

Oscillations near Peaks

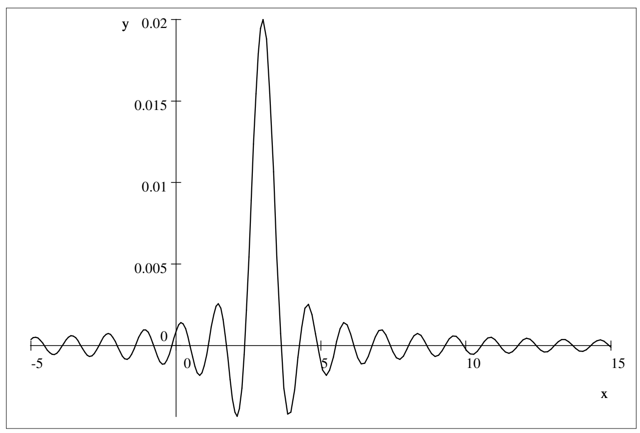
Often as a nonlinear system approaches an peak oscillations occur. A possible fit function might be

$$\boxed{\frac{Q_\infty}{n\tau}\frac{(2^n-1)\exp\left(\frac{t-a}{b}\right)}{\left[1+(2^n-1)\exp\left(\frac{t-a}{b}\right)\right]^{\frac{n+1}{n}}}+\frac{0.25}{2d(x-c-d)}\sin[2d(x-c-d)]}$$

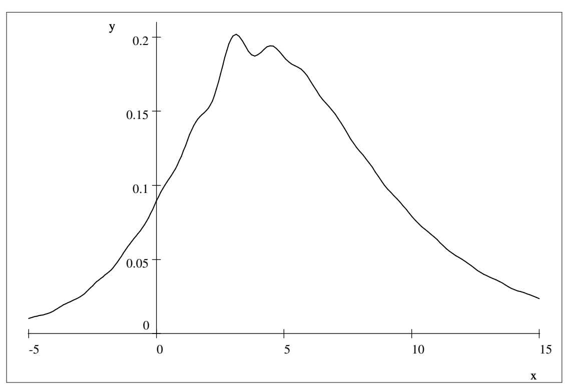
Plot $\frac{2}{2 \cdot 2} \frac{(2^2-1)\exp\left(\frac{x-5}{2}\right)}{\left[1+(2^2-1)\exp\left(\frac{x-5}{2}\right)\right]^{\frac{2+1}{2}}}$:



and $\frac{0.02}{5(x-3)} \sin(5(x-3))$:



Combined: $\frac{2}{2 \cdot 2} \frac{(2^2-1)\exp\left(\frac{x-5}{2}\right)}{\left[1+(2^2-1)\exp\left(\frac{x-5}{2}\right)\right]^{\frac{2+1}{2}}} + \frac{0.02}{5(x-3)} \sin(5(x-3))$:



A general equation might be

$$\frac{Q_\infty}{n\tau} \frac{(2^n-1)\exp\left(\frac{t-a}{b}\right)}{\left[1+(2^n-1)\exp\left(\frac{t-a}{b}\right)\right]^{\frac{n+1}{n}}} + \frac{g}{2e(x-f)} \sin[2e(x-f)]$$

Conclusion

Several functions that approach asymptotes and that peak are given that can be used to fit empirical data for nonlinear physical and sociological data. Some examples are:

- http://www.roperld.com/science/minerals/minerals.htm
- http://www.roperld.com/science/LifeExpectancyUS.htm
- http://www.roperld.com/science/GlobalWarmingRoper.htm
- http://www.roperld.com/science/EarthLimits.pdf
- http://arts.bev.net/roperldavid/majoriceages.htm
- http://www.roperld.com/science/NerveExcitation.pdf