Relativistic Kinematics of the Kerr-Model Ring Singularity in a Black-Hole

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The following discusses two situations for black-hole spins, according to the Kerr Model:

1. The spin as seen by an observer far outside the event horizon of a spinning black hole.
2. The spin as seen by an observer inside the event horizon of a spinning black hole.

An Observer Far Outside a Spinning Black Hole

To an observer far outside a black hole all the mass/energy that gets to the event horizon remains located at the event-horizon radius r_{EH}. So, the spin of the black hole as seen by an outside observer is the angular momentum of the total mass of the black hole circulating at the event-horizon equator radius.

(Masses in solar-mass units: $M_{\odot}$ = 1.989×10^{30} kg, G = 6.77408×10^{-11} m^3/kg/s^2, c = 2.99792458×10^8 m/sec.)

Angular momentum: $J = Mvr_{EH} \equiv acM \equiv \sigma \frac{GM^2}{c}$, where $0 \le \sigma \le 1$ so that r_{EH} is real (no naked singularity), where $v =$ circular velocity of the black-hole mass at the event-horizon radius, r_{EH}, as viewed by an observer at $r > r_{EH}$.

Event-horizon radius = $r_{EH} = \frac{1}{2}\left(r_S + \sqrt{r_S{}^2 - 4a^2}\right)$, where $r_S \equiv \frac{2GM}{c^2}$ and $a \equiv \frac{J}{cM}$ and $\sigma \equiv \frac{cJ}{GM^2} = \frac{c^2 a}{GM}$:

$$\therefore r_{EH} = \frac{1}{2}\left(\frac{2GM}{c^2} + \sqrt{\left(\frac{2GM}{c^2}\right)^2 - 4\left(\frac{J}{cM}\right)^2}\right) = \frac{1}{2}\left(\frac{2GM}{c^2} + \sqrt{\left(\frac{2GM}{c^2}\right)^2 - 4\left(\sigma\frac{GM}{c^2}\right)^2}\right) = .$$

$$r_{EH} = \frac{GM}{c^2}\left(1 + \sqrt{1-\sigma^2}\right) = \frac{1}{2} r_S \left(1 + \sqrt{1-\sigma^2}\right); 0 \le \sigma \le 1 \text{ for real } r_{EH}$$

The black-hole mass $= M = \gamma M_0$, where $M_0 \equiv$ black-hole rest mass and $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$, where $v =$ circular velocity of the black-hole mass at the event-horizon radius, r_{EH}.

Therefore $v = \pm c\frac{\sqrt{M^2 - M_0^2}}{M}$. Limits: $M_0 = M \Rightarrow v = 0$ and $M_0 = 0 \Rightarrow v = c$.

Generally, the circular speed is very high, c for photons.

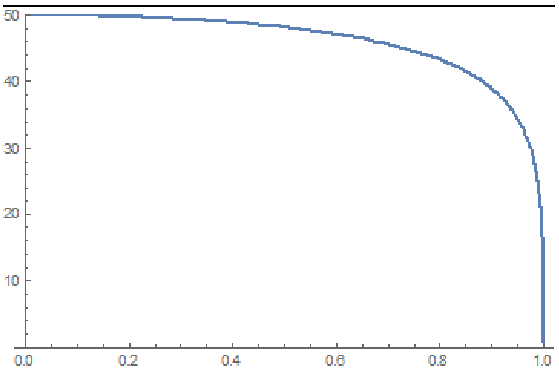
$$\therefore J \equiv \sigma\frac{GM^2}{c} = Mvr_{EH} = \sqrt{M^2 - M_0^2}\,\frac{GM}{c}\left(1 + \sqrt{1-\sigma^2}\right) \text{ where } 0 \le \sigma \le 1.$$

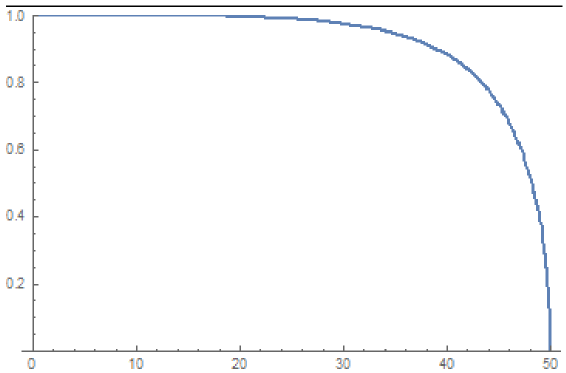
Finally, $\sigma M = \sqrt{M^2 - M_0^2}\left(1 + \sqrt{1-\sigma^2}\right)$.

Solve for $M_0 = \frac{\sqrt{2}M}{\sigma}\sqrt{\sigma^2 - 1 + \sqrt{1-\sigma^2}}$; solve for $\sigma = 2M\frac{\sqrt{M^2 - M_0^2}}{2M^2 - M_0^2}$.

(LIGO/O2 averages: $M = 50M_\odot$ and $\sigma = 0.7$.)

Plots of $M_0(\sigma)$ and $\sigma(M_0)$ for $M = 50M_\odot$:





LIGO/O2 remnant averages: $M = 50M_\odot$ and $\sigma = 0.7 : M_0 = 45.7M_\odot$.

Thus, the **average LIGO/O2 remnant black-hole mass on the event horizon is 91% rest mass**.

Suppose v = c: $J \equiv \sigma\frac{GM^2}{c} = Mv\,r_{EH} = \frac{GM^2}{c}\left(1 + \sqrt{1-\sigma^2}\right) \Rightarrow \sigma = 1 + \sqrt{1-\sigma^2} \Rightarrow \sigma = 1$. Same result for $M_0 = 0$.

For $M_0 = M \Rightarrow \sigma = 0$.

Inside a Spinning Black Hole According to Kerr Model

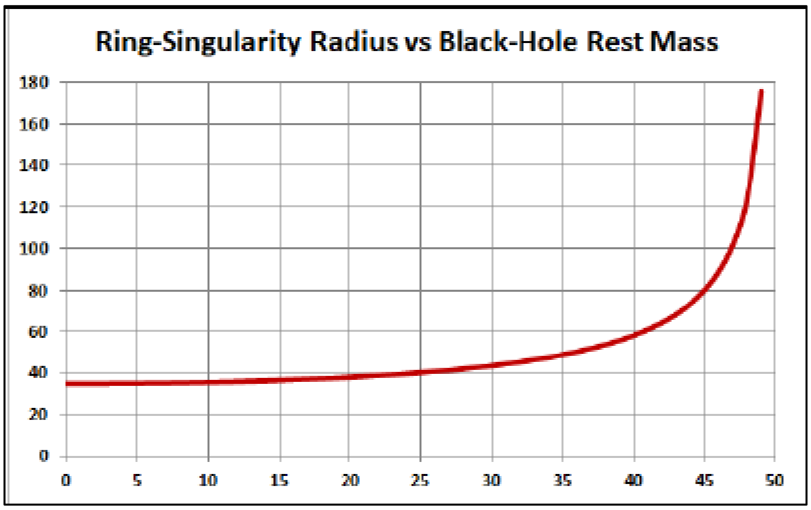
- Assume that a ring singularity of a black-hole (BH) is at radius r_c from the center of the spinning black hole (BH) of mass M_{BH}. According to the Kerr Model (Eq. 48) $r_c = a \equiv J / cM$ where $J \equiv$ angular momentum of the BH. Since no measurements can be made inside the event horizon (EH) of a BH, there is no possibility of measurable evidence that the Kerr Model is correct inside the EH. However, start with r_c to be determined by relativistic kinematics.
- Assume that the black-hole mass is known in solar-mass units ($M_\odot$ = 1.989×10^{30} kg) and **all of it is in the ring singularity**. (The average LIGO/O2 average remnant mass $M = 50M_\odot$ will be used as an example in this article.)
- Assume that relativistic kinematics applies to the ring singularity:
- From above $v = c\frac{\sqrt{M^2 - M_0^2}}{M}$. Limits: $M_0 = M \Rightarrow v = 0$ and $M_0 = 0 \Rightarrow v = c$.

  $M_0 = 0$ means that the ring singularity is entirely made of circulating photons and/or other zero-mass fundamental particles; perhaps zero mass because no Higgs field is present to give them mass.

- The ring-singularity spin is: $\boxed{J = Mvr_c = c\sqrt{M^2 - M_0^2}\, r_c}$. Then $\boxed{c\sqrt{M^2 - M_0^2}\, r_c = \sigma \frac{GM^2}{c}}$, where $\boxed{\sigma \equiv \frac{cJ}{GM^2}}$ is a dimensionless spin parameter σ for which $0 \le \sigma \le 1$; the upper limit prevents a naked singularity (r_{EH} is real).
   - For the **Kerr Model**: $J = Mva = MvJ/cM = vJ/c \Rightarrow \boxed{v = c \text{ and } M_0 = 0}$; i.e., the ring singularity is composed of photons and/or other zero-mass fundamental particles. Since the innermost orbit outside the event horizon (EH) is the prograde photon orbit (See Appendix.), it makes sense that only photons enter the EH.
- There are **two unknowns**, r_c and M_0, the radius of the ring singularity from the center of the BH and the rest mass of the ring singularity. Solve $c\sqrt{M^2 - M_0^2}\, r_c = \sigma \frac{GM^2}{c}$ for $\boxed{r_c = \frac{\sigma GM^2}{c^2\sqrt{M^2 - M_0^2}}}$.
- Solve $c\sqrt{M^2 - M_0^2}\, r_c = \sigma \frac{GM^2}{c}$ for $\boxed{M_0 = \frac{M}{c^2 r_c}\sqrt{c^4 r_c^2 - \sigma^2 G^2 M^2}}$. Note that $\boxed{M_0 = 0 \text{ when } r_c = \sigma \frac{GM}{c^2} = a}$ as described above.

  Use as an example: $\boxed{M = 50M_\odot \text{ and } \sigma = 0.7}$ for the average remnant BHs of LIGO-O2 binary BH mergers given in an appendix: $\boxed{r_c = a = \sigma \frac{GM}{c^2} = 35\frac{GM_\odot}{c^2} \text{ for } M_0 = 0}$.

Plot r_c vs M_0 with $\sigma = 0.7$ and $M = 50M_\odot$ in units $[r_c] = GM_\odot / c^2$:



Limits: $r_c = \sigma \frac{GM}{c^2} = 35\frac{GM_\odot}{c^2}$ for $M_0 = 0$ and $r_c = \infty$ for $M_0 = M$.

Compare to the equatorial event horizon:

$\boxed{r_{EH} = \frac{1}{2}\left(r_S + \sqrt{r_S^2 - 4a^2}\right)}$ where $r_S \equiv \frac{2GM}{c^2}$ and $\boxed{a \equiv \frac{J}{cM}}$ and $\boxed{\sigma \equiv \frac{cJ}{GM^2}}$:

$$\therefore r_{EH} = \frac{1}{2}\left(\frac{2GM_{BH}}{c^2} + \sqrt{\left(\frac{2GM}{c^2}\right)^2 - 4\left(\frac{J}{cM}\right)^2}\right) = \boxed{r_{EH} = \frac{GM}{c^2}\left(1+\sqrt{1-\sigma^2}\right)}$$

$$= \boxed{85.7\frac{GM_\odot}{c^2} = r_{EH} \text{ for } M = 50M_\odot \text{ and } \sigma{=}0.7} \text{ from LIGO-O2 averages.}$$

$$\text{If } r_c = r_{EH} = \frac{\sigma GM^2}{c^2\sqrt{M^2 - M_0^2}} = \frac{1750}{\sqrt{2500M_\odot - M^2{}_0}}\frac{GM_\odot}{c^2} = 85.7\frac{GM_\odot}{c^2} \Rightarrow M_0 = 45.7M_\odot$$

Of course, the event-horizon radius r_{EH} must be greater than the radius r_c of the ring singularity:

$\boxed{\text{For } r_c < r_{EH} = 85.7\frac{GM_\odot}{c^2} \Rightarrow M_0 < 45.7M_\odot}$ for a BH with $M = 50M_\odot$ and spin parameter $\sigma = 0.7$.

For a BH of mass $M = 50\text{M}_\odot$ and spin parameter $\sigma = 0.7$:

$\boxed{35\frac{GM_\odot}{c^2} \le r_c < 85.7\frac{GM_\odot}{c^2}}$ and $\boxed{0 \le M_0 < 45.7M_\odot}$.

Since about 99% of the mass of nucleons is binding energy of the constituent quarks, r_c and M_0 must be near or at the lower limit. The lower limit is the case for the Kerr Model:

The Kerr Model has a ring singularity at $r_c = a$. The average spin and BH mass for the LIGO-O2 binary BH mergers are $\sigma = 0.7$ and $M{=}50\text{M}_\odot$ where $\sigma = \frac{cJ}{GM^2}$ and $a = \sigma\frac{GM}{c^2}$. So, $\boxed{r_c = a = \sigma\frac{GM}{c^2} = 35\frac{GM_\odot}{c^2}}$ the lower limit for $M_0 = 0$ given above. So**, for the Kerr Model the mass of the ring-singularity is totally due to kinetic energy and the circular speed is** $v = c$ **!** The ring singularity could be just photons orbiting at speed c. In that case, apparently the “fundamental particles” quarks, electrons and electron neutrinos are crushed out of existence in the BH or never enter the event horizon! Perhaps some fundamental particles with rest mass are converted to photons in the accretion disk around the equator of the BH or are expelled from the vicinity of a BH in the BH jets.

Instead of there being a ring singularity, there is likely a finite but very small ring (a torus?) due to quantum effects of closely-packed, on a Planck scale, circulating photons. (The reference is a work in progress.)

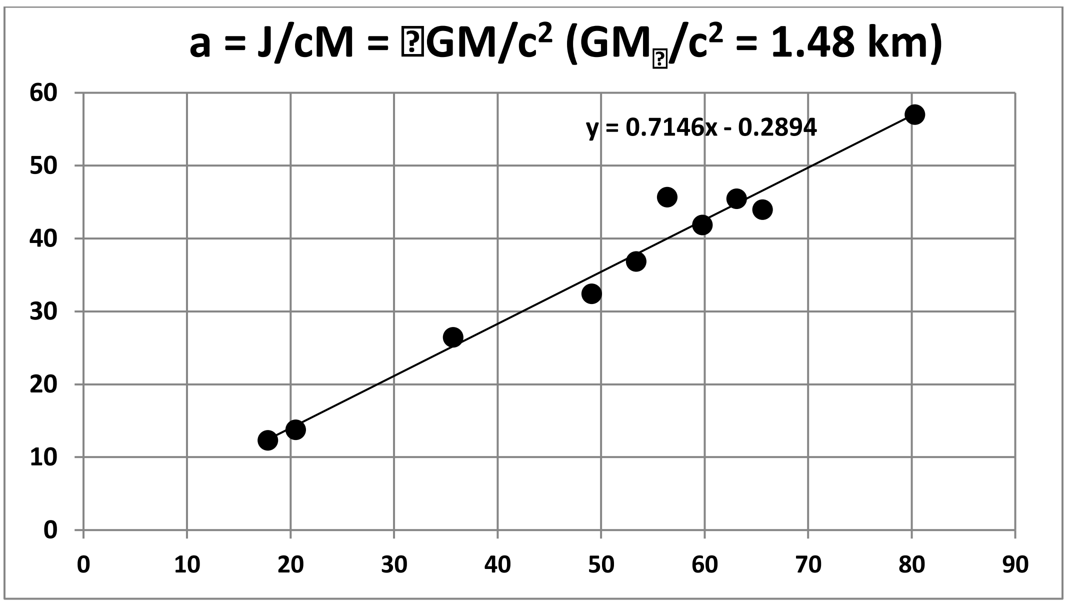
Appendix: LIGO-O2 Results

LIGO-O2 Remnant Spins $\sigma = \dfrac{cJ}{GM^2}$ and mass in solar masses **M** :

Binary BH	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Average**	**Median**	**Standard Deviation**
Remnant Mass (M)	**18**	**17.8**	**20.5**	**49.1**	**53.4**	**56.4**	**59.8**	**63.1**	**65.6**	**89.3**	**50.2**	**54.9**	**20.0**
Spin σ (cJ/GM^2)	**0.69**	**0.67**	**0.74**	**0.66**	**0.69**	**0.81**	**0.7**	**0.72**	**0.67**	**0.71**	**0.706**	**0.695**	**0.044**

Average spin parameter = 0.706, median spin parameter = 0.695; standard deviation = 0.044

For these LIGO remnant masses and spin parameters (σ) the ring-singularity radii are
$a = \sigma M$ in units $\left[GM_{\odot}/c^2\right]$**:**



Appendix: [Photon Orbits for Kerr Model of Black Holes](#)

The prograde photon orbit is in the equatorial plane of radius:

$$r_{p-} = \frac{2GM}{c^2}\left[1+\cos\left\{\frac{2}{3}\cos^{-1}\left(-|\sigma|\right)\right\}\right] \text{ where } \sigma \equiv \frac{cJ}{GM^2}.$$

$$\text{The range is } \boxed{\frac{GM}{c^2} \le r_{p-} \le \frac{3GM}{c^2} \text{ for } 1 \ge \sigma \ge 0}.$$

$$\text{There is a retrograde photon orbit farther out: } r_{p+} = \frac{2GM}{c^2}\left[1+\cos\left\{\frac{2}{3}\cos^{-1}\left(|\sigma|\right)\right\}\right]$$

$$\text{The range is } \frac{3GM}{c^2} \le r_{p+} \le \frac{4GM}{c^2} \text{ for } 0 \le \sigma \le 1.$$

$$\text{Compare to the event horizon: } r_{EH} = \frac{GM}{c^2}\left(1+\sqrt{1-\sigma^2}\right) \Rightarrow \boxed{\frac{GM}{c^2} \le r_{EH} \le 2\frac{GM}{c^2} \text{ for } 0 \le \sigma \le 1}.$$

For $M = 50M_{\odot}$ and $\rho = 0.7$ the LIGO-O2 averages:

$$\boxed{r_{EH} = 85.7\frac{GM_{\odot}}{c^2} \text{ and } r_{p-} = 100.7\frac{GM_{\odot}}{c^2}}.$$

Appendix: Compare Kerr-Black-Hole Equatorial Boundaries and Orbits

Event Horizon:

https://en.wikipedia.org/wiki/Kerr_metric#Important_surfaces

$$r_{EH} = \left(1+\sqrt{1-\sigma^2}\right)$$

Ergosphere Boundary

https://en.wikipedia.org/wiki/Kerr_metric#Ergosphere_and_the_Penrose_process:

$$r_{ES} = \left(1+\sqrt{1-\sigma^2\cos\theta}\right) \xrightarrow[Equator]{} 2, \text{ where } \theta = \pi/2 \text{ at the equator.}$$

Innermost Photon Orbit

https://www.physics.nus.edu.sg/~phyteoe/kerr/paper.pdf

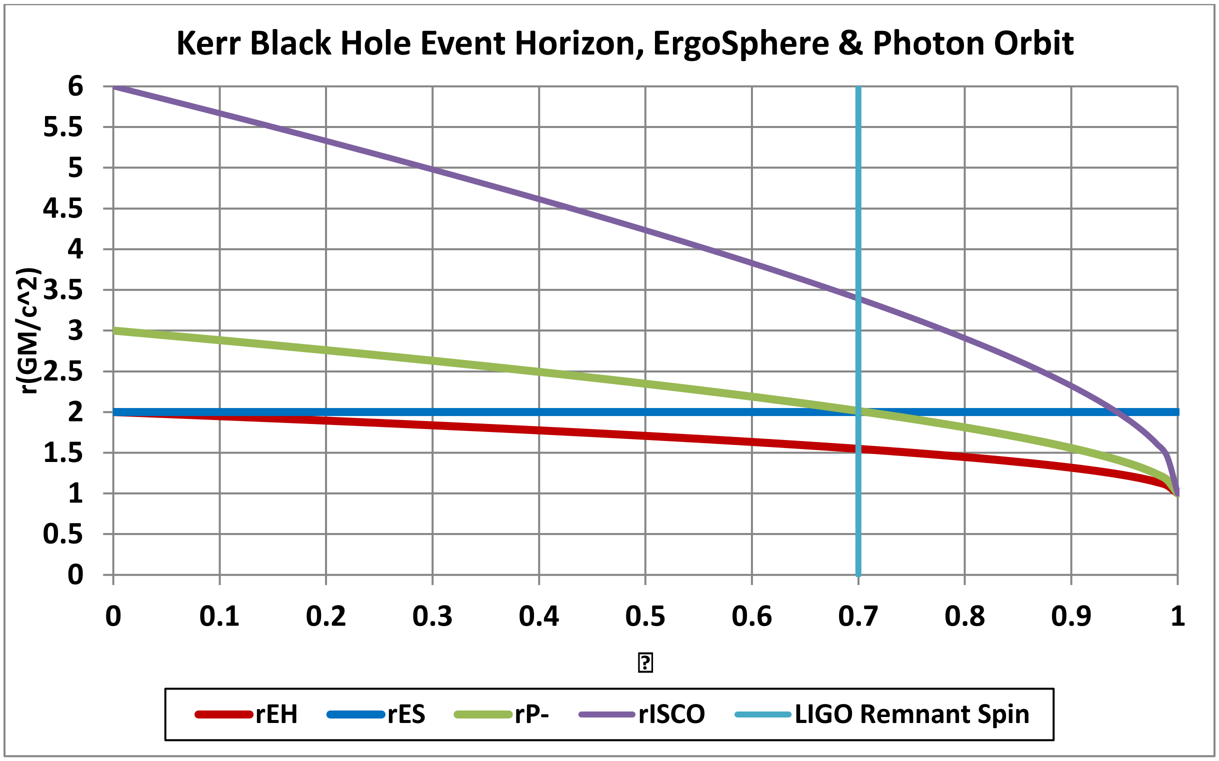
$$r_{p-} = 2\left[1+\cos\left\{\frac{2}{3}\cos^{-1}\left(-|\sigma|\right)\right\}\right] \xrightarrow[\sigma=1/\sqrt{2}=0.707]{} 2$$

Innermost Stable Circular Orbit (ISCO) for a Particle-with-Mass

https://en.wikipedia.org/wiki/Innermost_stable_circular_orbit:

$$r_{ISCO} = 3 + n + \sqrt{(3-m)(3+m+2n)}$$

where $m \equiv 1 + \sqrt[3]{1-\sigma^2}\left(\sqrt[3]{1+\sigma} + \sqrt[3]{1-\sigma}\right)$ and $n \equiv \sqrt{3\sigma^2 + m^2}$.



Is it important that the inner photon orbit radius and the Ergosphere equatorial radius are equal when σ = 0.7, the average spins of the LIGO-O2 remnant masses?

Appendix: Ring Singularity Mathematics

The most complete treatment of the mathematics of the Kerr-Model ring singularity I have found is The Kerr spacetime: A brief introduction by Matt Visser:

In Kerr-Schild coordinates (Eq. 48):

$$x^2 + y^2 = a^2,\ z = 0 \text{ where } a \equiv \frac{J}{Mc} \equiv \sigma \frac{GM}{c^2}.$$

Appendix: Kerr-Model Event Horizon Mathematics

The 3-dimensional shape of the event horizon is (Eq. 126):

$$\boxed{x^2 + y^2 + 2\frac{GM}{c^2 r_{EH}} z^2 = 2\frac{GM}{c^2} r_{EH}} \text{ where } r_{EH} = \frac{GM}{c^2} + \sqrt{\frac{G^2M^2}{c^4} - a^2} = \frac{GM}{c^2}\left(1 + \sqrt{1-\sigma^2}\right)$$

$$\Rightarrow c^2 r_{EH}\left(x^2 + y^2\right) + 2GMz^2 = 2GMr_{EH}^2 \Rightarrow \boxed{\left(1+\sqrt{1-\sigma^2}\right)\left(x^2+y^2\right) + 2z^2 = 2\left(\frac{GM}{c^2}\right)^2 \left(1+\sqrt{1-\sigma^2}\right)^2}.$$

Appendix: Kerr-Model Ergosphere Mathematics

The 3-dimensional shape of the ergosphere is (Eq. 155):

$$\sqrt{x^2+y^2} = \left[\left(\frac{GM}{c^2} + \sqrt{\left\{\frac{GM}{c^2}\right\}^2 - a^2\cos^2\theta}\right)^2 + a^2\right]\sin\theta \text{ and } z = \left(\frac{GM}{c^2} + \sqrt{\left\{\frac{GM}{c^2}\right\}^2 - a^2\cos^2\theta}\right)\cos\theta.$$

$$\Rightarrow \boxed{\sqrt{x^2+y^2} = \frac{GM}{c^2}\left[\left(1+\sqrt{1-\sigma^2\cos^2\theta}\right)^2 + \sigma^2\right]\sin\theta} \text{ and } \boxed{z = \frac{GM}{c^2}\left(1+\sqrt{1-\sigma^2\cos^2\theta}\right)\cos\theta}$$