Introduction

Newton’s Law of Gravity (NLG, http://en.wikipedia.org/wiki/Newton_law_of_gravity) for the force between two masses, \( F(r) = -G \frac{m_1 m_2}{r^2} \), predicts that the expansion of the universe due to the Big Bang (http://en.wikipedia.org/wiki/Big_bang) should be slowing down. The minus sign indicates that the mutual force is attractive. However, it is known that the expansion of the universe is accelerating (http://en.wikipedia.org/wiki/Accelerating_universe), at least for large separation distances. NLG is very accurate on the Earth and in the solar system and has been so for very long times, so any deviation from NLG must be small for small separations and short times compared to the age of the universe, \((13.798 \pm 0.037) \times 10^9\) years (http://en.wikipedia.org/wiki/Age_of_universe).

Although NLG has been replaced by the much more complicated, both mathematically and conceptually, General Theory of Relativity (GTR, http://en.wikipedia.org/wiki/General_theory_of_relativity), it may have some pedagogical value for beginning physics/astronomy students and other interested persons to consider a Newtonian view of the accelerating expansion of the universe. Of course, the Newtonian view is not as accurate as the GTR.

The NLG potential energy between two masses is defined as \( U(r) = -\int F(r) \, dr = \int G \frac{m_1 m_2}{r^2} \, dr = -G \frac{m_1 m_2}{r} \).

Consider a “Simple Universe” consisting of two bodies with identical masses \( m \) that are separated by distance \( r_0 \) and moving with relative velocity \( v_0 \) at some time \( t_0 \) caused by the Big Bang. This article will consider possible changes to the NLG to account for the accelerating expansion of this Simple Universe.

This article calculates the relative velocity and relative acceleration of the two equal masses in this simplest of universes for several assumptions about changes to the NLG to achieve an accelerating universe. It uses the two-body energy equation (TBEQ) given in http://vv-voronkov.spb.ru/EN/two-lawe.html:

\[
\frac{1}{2} m v_i^2 + \frac{m_k}{m_i + m_k} U_{ik} = C_i
\]

where \( i = 1, 2 \), \( k = 2, 1 \) and \( U_{ik} \) = potential energy of body \( k \) at location of body \( i \).

The positions and velocities are relative to the center of mass of the two bodies.

For equal masses \( m_1 = m_2 = m \) the TBEQ is \( mv^2 + U_k = 2C_i \equiv c_i \).

The relative velocity, \( v(r) = v_1(r) + v_2(r) \) can be calculated from the TBEQ for a given \( U(r) \).

The relative acceleration can be calculated from \( a(r) = a_1(r) + a_2(r) = 2 \frac{F(r)}{m} \).
Newton's Deceleration of the Expansion of the Universe

For the Simple Universe the NLG potential energy is

\[ U_{ik}(r_{ik}) = -G \frac{m^2}{r_{ik}} \quad \text{or} \quad U(r) = -G \frac{m^2}{r} \quad \text{where} \quad r_{ik} = r_i + r_k \equiv r \]

is the separation of the two bodies.

The mutual force between the two bodies is

\[ F(r) = -\frac{dU(r)}{dr} = -G \frac{m^2}{r^2} \]

Therefore, the TBEQ is \[ v_1^2 = \frac{mG}{r} = v_{10}^2 \frac{mG}{r_0^2} \]

or \[ v_1^2 = v_{10}^2 + mG \left( \frac{1}{r} - \frac{1}{r_0} \right) = v_{10}^2 - mG \left( \frac{r - r_0}{rr_0} \right) \]

or \[ v_1 = \sqrt{v_{10}^2 - mG \left( \frac{r - r_0}{rr_0} \right)} \].

Due to the two bodies having the same mass, \( m \), the two velocities are always opposite and the same magnitude. Thus, the relative velocity is \( v = v_1 + v_2 = 2v_1 = 2v_2 \). Thus, \( v_{10} = v_{20} = \frac{v_0}{2} \).

The relative velocity as a function of \( r \) is \( v(r) = v_1 + v_2 = 2 \sqrt{\left( \frac{v_0}{2} \right)^2 - mG \left( \frac{r - r_0}{rr_0} \right)} \) or

\[ v(r) = \sqrt{v_{10}^2 + 4mG \left( \frac{1}{r} - \frac{1}{r_0} \right)} \].

The relative velocity will be zero when \( r r_0 v_{10}^2 = 4mG (r_i - r_0) \), where \( r_i \) is the turn-around separation where the velocity changes direction; thus, \[ r_i = \frac{4mG r_0}{4mG - r_0 v_{10}^2} \].

Then the relative velocity will reverse direction from outward to inward and the expansion will become contraction between the two masses.

If \( v_0 = \sqrt{\frac{4mG}{r_0}} \) then \( r_i = \infty \),

for this case the relative velocity of the Simple Universe never changes sign but always decreases.
The relative acceleration is \( a(r) = 2 \frac{F(r)}{m} = -\frac{2mG}{r^2} \), which is always negative.

The NLG causes a deceleration of the universe. To have an accelerating universe a positive term can be added to the mutual force for the two bodies.

**Simplest Modification of NLG to Achieve an Accelerating Universe (Constant Force Added)**

The simplest modification to NLG to have an accelerating universe is to add a small positive constant repulsive (positive) force to NLG:

\[
F(r) = -G \frac{m^2}{r^2} + Gf^2, \quad \text{then} \quad U(r) = -\int F(r)dr = -G \frac{m^2}{r} - Gf^2 r.
\]

Example for Simple Universe: left: add positive constant force to NLG versus separation \( r \); right: add negative linear potential energy to NLG versus separation \( r \):

![Graphs showing force and potential energy](image)

Acceleration is essentially the same curve as the force.

If the initial kinetic energy is large enough to get over the potential barrier, the Simple Universe will expand with increasing speed.

The repulsive force will be larger than the NLG force at the potential-energy peak and the separation, \( r_p \), where the force goes through 0: \( r_p = \frac{m}{f} \).

Then, for the Simple Universe of two equal masses the TBEQ is \( v_1^2 = -\frac{mG}{r} - G \frac{f^2}{m} r = v_{10}^2 - \frac{mG}{r_0} - G \frac{f^2}{m} r_0 \).

Then \( v_1^2 = v_{10}^2 + mG \left( \frac{1}{r} - \frac{1}{r_0} \right) + G \frac{f^2}{m} (r - r_0) \) or \( v_1 = \sqrt{v_{10}^2 - mG \left( \frac{r - r_0}{r_0} \right) + G \frac{f^2}{m} (r - r_0)} \).

The relative velocity as a function of \( r \) is

\[
v(r) = v_1 + v_2 = \sqrt{v_{10}^2 - 4mG \left( \frac{r - r_0}{rr_0} \right) + 4G \frac{f^2}{m} (r - r_0)} \quad \text{where} \quad v_0 = v_{10} + v_{20} = 2v_{10} = 2v_{20},
\]
or \( v(r) = \sqrt{v_0^2 - 4G (r - r_0) \left( \frac{m}{rr_0} - \frac{f^2}{m} \right)} \).

For the Simple Universe to always expand rather than eventually contract the initial relative velocity must satisfy:

\[ v_0^2 > 4G (r_p - r_0) \left( \frac{m}{r_p r_0} - \frac{f^2}{m} \right) \]

or \( v_0^2 > \frac{4G}{m} (m - fr_0)^2 \) using \( r_p = \frac{m}{f} \).

The relative acceleration is

\[ a(r) = 2 \frac{F(r)}{m} = -2G \frac{m}{r^2} + 2G \frac{f^2}{m} = 2G \frac{f^2 r^2 - m^2}{mr^2} \]

The acceleration is negative (deceleration) for \( r < \frac{m}{f} \) and positive for \( r > \frac{m}{f} \). Thus, the expansion of the Simple Universe can accelerate.

**Cosmological-Constant Modification of NLG to Achieve an Accelerating Universe (Linear Force Added)**

Eq. 2-6 of the article “The Consistent Newtonian Limit of Einstein’s Gravity with a Cosmological Constant” (http://arxiv.org/pdf/gr-qc/0004037.pdf and p.8 in http://www.fis.utfsm.cl/hep2008/files/nowakowski.pdf). purports to be the Newtonian limit for the General Theory of Relativity (GTR: See the Appendix.) in terms of the potential. For the Simple Universe the potential energy and force are:

\[ U(r) = -G \frac{m^2}{r} - \frac{m}{6} \Lambda r^2 \text{ and } F(r) = -\frac{dU(r)}{dr} = -G \frac{m^2}{r^2} + \frac{m}{3} \Lambda r \]

The added force term could be labeled a “repulsive Hooke’s Law” (linear in r) (http://en.wikipedia.org/wiki/Hooke%27s_law). This could be thought of as a two simple springs attaching both masses to opposite infinities.

Example for Simple Universe: left: add linear force to NLG versus separation, r; right: add quadratic potential energy to NLG versus separation, r:
This is the Newtonian limit of adding the cosmological constant (http://en.wikipedia.org/wiki/Cosmological_constant) to the GTR, which is interpreted as “Dark Energy” (http://en.wikipedia.org/wiki/Dark_energy).

See also http://www.fis.utfsm.cl/hep2008/files/nowakowski.pdf: ‘The last term plays the role of a repulsive external force! The Galilean spacetime gets replaced by Newton-Hooke spacetime where each two space points go apart due to the cosmological constant. This is the part of the cosmological expansion which survives the Newtonian limit’.

The repulsive force will be larger than the NLG force at the potential-energy peak and the separation, $r_p$, where the force goes through 0: $r_p = \sqrt[3]{\frac{3mG}{\Lambda}}$.

Then, for the Simple Universe of two equal masses the TBEQ is $v_1^2 - \frac{mG}{r} - \frac{\Lambda}{6} r^2 = v_{10}^2 - \frac{mG}{r_0} - \frac{\Lambda}{6} r_0^2$.

Then $v_1^2 = v_{10}^2 + mG\left(\frac{1}{r} - \frac{1}{r_0}\right) + \frac{\Lambda}{6} (r^2 - r_0^2)$ or $v_1 = \sqrt{v_{10}^2 - mG\left(\frac{r-r_0}{rr_0}\right) + \frac{\Lambda}{6} (r^2 - r_0^2)}$

The relative velocity as a function of $r$ is

$v(r) = v_1 + v_2 = \sqrt{v_0^2 - 4mG\left(\frac{r-r_0}{rr_0}\right) + \frac{2\Lambda}{3} (r^2 - r_0^2)}$ where $v_0 \equiv v_{10} + v_{20} = 2v_{10} = 2v_{20}$,

or $v(r) = \sqrt{v_0^2 -(r-r_0) \left[\frac{4mG}{rr_0} - \frac{2\Lambda}{3} (r + r_0)\right]}$

For the Simple Universe to expand instead of contract the initial relative velocity and position must satisfy:

$v_0^2 > 2\left(\frac{2mG}{r_p r_0} - \frac{\Lambda}{3}(r_p^2 - r_0^2)\right)$

The relative acceleration is $a(r) = \frac{F(r)}{m} = \frac{\Lambda r^3 - 3Gm}{3r^2}$

The acceleration will be negative (deceleration) for $r^3 < \frac{3Gm}{\Lambda}$ and positive for $r^3 > \frac{3Gm}{\Lambda}$. Thus, the expansion of the Simple Universe can accelerate.
**Exponential Force Added**

Adding a small positive constant or linear repulsive force to NLG may be too simple to correspond to reality. Perhaps the added force should be exponential with origin at some transition separation \( r_t \) of the masses:

\[
F(r) = -GM^2/r^2 + Gf^2 \exp \left( \frac{r - r_t}{\tau} \right)
\]

Then
\[
U(r) = -GM^2/r - Gf^2 \tau \exp \left( \frac{r - r_t}{\tau} \right)
\]

Example for Simple Universe: left: add exponential to NLG versus separation, \( r \); right: add exponential potential energy to NLG versus separation, \( r \):

Acceleration is essentially the same curve as the force.

If the initial kinetic energy is large enough to get over the potential barrier, the Simple Universe will expand with increasing speed.

The repulsive force will be larger than the NLG force at the potential-energy peak and the separation, \( r_p \), where the force goes through 0:

\[
\frac{m^2}{r_p^2} = f^2 \exp \left( \frac{r_p - r_t}{\tau} \right)
\]

Then, for the Simple Universe of two equal masses the TBEQ is

\[
v_1^2 = \frac{mG}{r} - Gf^2 \tau \exp \left( \frac{r - r_t}{\tau} \right) = v_{10}^2 - \frac{mG}{r_0} - Gf^2 \tau \exp \left( \frac{r_0 - r_t}{\tau} \right)
\]

\[
= v_{10}^2 + mG \left( \frac{1}{r} - \frac{1}{r_0} \right) + Gf^2 \tau \exp \left( \frac{r - r_t}{\tau} \right) - Gf^2 \tau \exp \left( \frac{r_0 - r_t}{\tau} \right)
\]

Then

or

\[
v_1 = \sqrt{v_{10}^2 - mG \left( \frac{r - r_0}{rr_0} \right) + Gf^2 \tau \exp \left( \frac{r - r_t}{\tau} \right) - Gf^2 \tau \exp \left( \frac{r_0 - r_t}{\tau} \right)}
\]

The relative velocity as a function of \( r \) is

\[
v(r) = v_1 + v_2 = \sqrt{v_0^2 - 4mG \left( \frac{r - r_0}{rr_0} \right) + 4Gf^2 \tau \left[ \exp \left( \frac{r - r_t}{\tau} \right) - \exp \left( \frac{r_0 - r_t}{\tau} \right) \right]}
\]
where \( v_0 \equiv v_{10} + v_{20} = 2v_{10} = 2v_{20} \).

For the Simple Universe to always expand rather than eventually contract the initial relative velocity must satisfy:

\[
v_0^2 > 4mG \left( \frac{r_p - r_0}{r_p r_0} \right) - 4Gf^2 \tau \left[ \exp \left( \frac{r_p - r_\tau}{\tau} \right) - \exp \left( \frac{r_0 - r_t}{\tau} \right) \right].
\]

The relative acceleration is

\[
a(r) = 2 \frac{F(r)}{m} = -2Gm \frac{m}{r^2} + 2Gf^2 \frac{f}{m} \exp \left( \frac{r - r_\tau}{\tau} \right).
\]

The acceleration is negative (deceleration) for \( \frac{m^2}{r^2} > f^2 \exp \left( \frac{r - r_\tau}{\tau} \right) \) and positive for \( \frac{m^2}{r^2} < f^2 \exp \left( \frac{r - r_\tau}{\tau} \right) \).

Thus, the expansion of the Simple Universe can accelerate.

**A State-Change Modification of NLG to Achieve an Accelerating Universe**

Adding a small positive constant or linear repulsive force to NLG may be too simple to correspond to reality. Perhaps the added force should be more like a state change from 0 to some asymptotic value centered at some specific transition separation \( r_t \) of the masses, e.g., as represented by the hyperbolic tangent:

Then

\[
F(r) = -Gm \frac{m^2}{r^2} + \frac{Gf^2}{2} \left[ 1 + \tanh \left( \frac{r - r_\tau}{\tau} \right) \right].
\]

Then

\[
U(r) = -\int F(r) \, dr = -Gm \frac{m^2}{r} - \frac{Gf^2}{2} \left\{ \tau \ln \left[ \exp \left( \frac{2r - r_\tau}{\tau} \right) + 1 \right] + 2r_t \right\}.
\]

The constant potential energy term \( -Gf^2 r_t \) can be dropped to yield

\[
U(r) = -Gm \frac{m^2}{r} - \frac{Gf^2}{2} \ln \left[ \exp \left( \frac{2r - r_\tau}{\tau} \right) + 1 \right].
\]

Example for Simple Universe: left: add force change of state to NLG versus separation, \( r \); right: add potential energy due to force change of state to NLG versus separation, \( r \):

Acceleration is essentially the same curve as the force. If the initial kinetic energy is large enough to get over the potential barrier, the Simple Universe will expand with increasing speed.
The repulsive force will be larger than the NLG force at the potential-energy peak and the separation, \( r_p \), where the force goes through 0:

\[
m^2 \frac{r_p^2}{r_p^2} = \frac{f^2}{2} \left[ 1 + \tanh \left( \frac{r_p - r}{r} \right) \right]
\]

Then, for the Simple Universe of two equal masses,

\[
v_1^2 - \frac{mG}{r} - \frac{Gf^2 \tau}{2m} \ln \left\{ \exp \left( 2 \frac{r - r_i}{r} \right) + 1 \right\} = v_1^2 - \frac{mG}{r_0} - \frac{Gf^2 \tau}{2m} \ln \left\{ \exp \left( 2 \frac{r_0 - r_i}{r} \right) + 1 \right\}.
\]

Then \( v_1^2 = v_{10}^2 + mG \left( \frac{1}{r} - \frac{1}{r_0} \right) + \frac{Gf^2 \tau}{2m} \ln \left\{ \exp \left( 2 \frac{r - r_i}{r} \right) + 1 \right\} - \ln \left\{ \exp \left( 2 \frac{r_0 - r_i}{r} \right) + 1 \right\} \]

The relative velocity as a function of \( r \) is

\[
v(r) = v_1 + v_2 = \sqrt{v_{0}^2 - 4mG \left( \frac{r - r_0}{r_0} \right) + \frac{2Gf^2 \tau}{m} \ln \left\{ \exp \left( 2 \frac{r - r_i}{r} \right) + 1 \right\} - \ln \left\{ \exp \left( 2 \frac{r_0 - r_i}{r} \right) + 1 \right\}}.
\]

where \( v_0 \equiv v_{10} + v_{20} = 2v_{10} = 2v_{20} \).

For the Simple Universe to expand instead of contract the initial relative velocity and position must satisfy:

\[
v_0^2 > 4mG \left( \frac{r - r_0}{rr_0} \right) - \frac{2Gf^2 \tau}{m} \ln \left\{ \exp \left( 2 \frac{r - r_i}{r} \right) + 1 \right\} - \ln \left\{ \exp \left( 2 \frac{r_0 - r_i}{r} \right) + 1 \right\}
\]

The relative acceleration is

\[
a(r) = \frac{F(r)}{m} = -2G \frac{m}{r^2} + \frac{Gf^2}{m} \left[ 1 + \tanh \left( \frac{r - r_i}{r} \right) \right]
\]

The acceleration will be negative (deceleration) for \( 2m^2 > f^2 r^2 \left[ 1 + \tanh \left( \frac{r - r_i}{r} \right) \right] \) and positive for \( 2m^2 < f^2 r^2 \left[ 1 + \tanh \left( \frac{r - r_i}{r} \right) \right] \). Thus, the expansion of the Simple Universe can accelerate.

**Conclusion and Questions**

- The cosmological constant in GTR is equivalent to adding a repulsive force term to the NLG that is linear in the separation of two bodies. Is the acceleration of the universe really that simple, or should the cosmological constant be replaced by a cosmological variable?
- If the added term to the NLG is a constant, \( C \), the radial dependence of the cosmological term would presumably be \( \Lambda (r) = \frac{3C}{r} \). A GTR expert is needed to translate this into the GTR equation.
If the added term to the NLG is an exponential force, the radial dependence of the cosmological term would presumably be \( \Lambda(r) = \frac{3C}{r} \left[ \exp\left( \frac{r - r_s}{\tau} \right) \right] \). A GTR expert is needed to translate this into the GTR equation.

If the added term to the NLG is a transitional force represented by the hyperbolic tangent, the radial dependence of the cosmological term would presumably be \( \Lambda(r) = \frac{3C}{2r} \left[ 1 + \tanh\left( \frac{r - r_s}{\tau} \right) \right] \). A GTR expert is needed to translate this into the GTR equation.

Simple-Universe example for velocities versus separation, \( r \), for four different terms added to NLG:

<table>
<thead>
<tr>
<th>Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

Appendix


The cosmological constant \( \Lambda \) appears in Einstein's field equation ([http://en.wikipedia.org/wiki/Einstein_field_equations](http://en.wikipedia.org/wiki/Einstein_field_equations)) in the form of

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},
\]

where \( R \) and \( g \) describe the structure of spacetime ([http://en.wikipedia.org/wiki/Spacetime](http://en.wikipedia.org/wiki/Spacetime)), \( T \) pertains to matter and energy affecting that structure, and \( G \) and \( c \) are conversion factors that arise from using traditional units of measurement. When \( \Lambda \) is zero, this reduces to the original field equation of general relativity. When \( T \) is zero, the field equation describes empty space (the vacuum [http://en.wikipedia.org/wiki/Spacetime](http://en.wikipedia.org/wiki/Spacetime)).