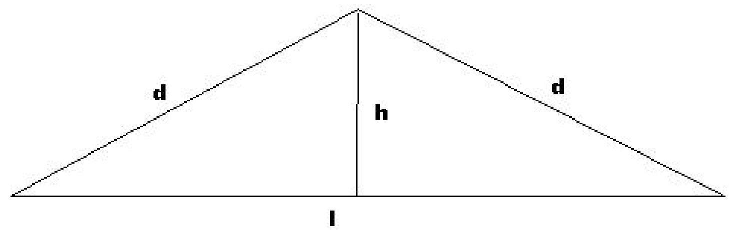
Energy Differences in Hilly Driving for Hybrid Vehicles:

Introduction

Consider the idealized hill (an isosceles triangle) shown in the box, with the left and right angle being θ .



Let d = sides of the triangle and l = the base of the triangle = flat distance traveled by a vehicle. Then $\boxed{h = \frac{l}{2}\tan\theta}$= the perpendicular distance of the peak to the base. Note that $\boxed{\frac{l}{2} = d\cos\theta}$.

Consider the case of **a hybrid vehicle traveling up and down the hill a distance** $2d$ **at a constant speed** v and compare it to the case of **a hybrid vehicle traveling on the flat a distance** l **from one side of the triangle to the other**.

The hybrid vehicle's energy system, which includes energy from both gasoline and an electric battery, moves the vehicle up the hill. Assume that no gasoline or battery energy is used to provide energy to move the vehicle **down** the hill. Some fraction f of the available energy when the vehicle goes down the hill is converted by a generator into electrical energy to charge the battery. (Even going up a hill that is not too steep, in reality some gasoline may be used to charge the battery. That will be neglected in this study.)

On the flat some gasoline may be used to charge the battery and some battery energy may be used to move the vehicle, depending on the state of charge of the battery.

The **gravitational potential energy gained** going from the bottom to the top of the hill is $\boxed{E_g = mgh = mg\frac{l}{2}\tan\theta}$ and the same amount of gravitational potential energy is available going from the top to the bottom of the hill to convert into heat energy or charge the battery.

Case of No Road Friction and No Air Drag

Traveling Over the Hill

Assume that there is no road friction nor air drag. The vehicle's energy system has to provide energy that is equal to the gravitational potential energy, $mg\frac{l}{2}\tan\theta$, to get the car to the top of the hill at constant speed. Since the vehicle is a hybrid, when going down the hill some fraction f of the gravitational potential energy can be converted to electrical energy by a generator to charge the vehicle's battery. The energy stored in the battery is $\boxed{E_s = fE_g = fmg\frac{l}{2}\tan\theta\text{, where } 0 < f < 1}$. The remaining gravitational potential energy, $E_r = (1-f)E_g$, must be dissipated by some internal vehicle friction, mostly by friction or engine braking, to keep the speed constant when going down the hill.

The gasoline engine is turned off and the battery does not supply any electrical energy when going down the hill. So the total energy supplied by the vehicle's energy system when traversing the entire hill is the energy converted to gravitational potential energy going up the hilll: $E = E_g = mg\frac{l}{2}\tan\theta > 0$. The energy-per-distance (edt) traveled is

$$\boxed{\text{edt} = \frac{mg\frac{l}{2}\tan\theta}{2d} = \frac{1}{2}mg\cos\theta\tan\theta = \frac{1}{2}mg\sin\theta} .$$

This is related to what the gasoline-usage display of the vehicle shows.

The energy-per-distance (edtf) traveled going over the hill relative to the straight line between the beginning and ending points of the hill (on the flat) is

$$\boxed{\text{edtf} = \frac{mg\frac{l}{2}\tan\theta}{l} = \frac{1}{2}mg\tan\theta} .$$

This is probably the better quantity to compare to traveling on the flat, if the goal is to travel the horizontal distance l.

If one assumes that the battery supplies the same amount of energy to move the vehicle up the hill as is put back into it when going down the hill, the total energy supplied by the gasoline to traverse the entire hill is $E = (1-f)E_g = (1-f)mg\frac{l}{2}\tan\theta$. Then the gasoline-energy-per-distance (edt) traveled is

$$\boxed{\text{edt(gasoline)} = \frac{(1-f))mg\frac{l}{2}\tan\theta}{2d} = \frac{1}{2}(1-f)mg\sin\theta} .$$

This is related to what the gasoline-usage display of the vehicle shows.

The gasoline-energy-per-distance (edtf) traveled going over the hill relative to the straight line between the beginning and ending points of the hill (on the flat) is

$$\boxed{\text{edtf(gasoline)} = \frac{(1-f)mg\frac{l}{2}\tan\theta}{l} = \frac{1}{2}(1-f)mg\tan\theta} .$$

This is probably the better quantity to compare to traveling on the flat, if the goal is to travel the horizontal distance l.

Traveling on the Flat

Since we assume here that there is no road friction nor air drag nor internal system friction, traveling at constant speed on the flat from one side of the triangle to the other causes no energy loss. So, in this case, edt = edtf = 0. That is, without friction, no energy is required to keep the vehicle moving at constant speed (Newtons First Law of mechanics). For a hybrid vehicle, however, some gasoline energy may be used to charge the battery.

Comparison of Hill to Flat Travel

So, edt/edft for going over a hill is larger than edt/edtf for traveling on the flat for the unrealistic case of no friction.

Case of Air Drag

Traveling Over the Hill

More realistically, assume that the only external frictional force of consequence is the **air drag due to the velocity v of the vehicle**. (For high speeds this is a good approximation for the external frictional forces.) Then the drag force is

$$\boxed{F = -\tfrac{1}{2}c_d A\rho v^2 \equiv -mCv^2}$$

(http://en.wikipedia.org/wiki/Drag_equation),

where

- c_d = drag coefficient (unitless) (0.26 for 2004-5 Toyota Prius II),
- A = projected area perpendicular to the velocity (~2.52 meter^2 for Toyota Prius II)
- ρ = air density $\approx 1.2\frac{\mathrm{kg}}{\text{meter}^3}$ (Depends on the temperature, air pressure and the amount of water vapor in the air.),
- m = mass of the vehicle (1311 kg for Toyota Prius II) (The mass is only needed in the equation directly below; it will cancel out in the final equations of interest.).
- $C \equiv \frac{1}{2}c_d A\rho/m$ (~$6.0 \times 10^{-4}\frac{1}{\text{meter}}$ for Toyota Prius II)

Since the speed remains constant going up the hill, the vehicle's energy system must provide the energy lost to heat by the **air drag going up the hill** at speed v, which is

$$\boxed{E_{ud} = Fd = mCv^2 d = mCv^2\frac{l}{2\cos\theta}}\ .$$

Similarly, gravity provides the energy lost to heat by **air drag going down the hill** at constant speed v, which is

$$\boxed{E_{dd} = Fd = mCv^2 d = mCv^2\frac{l}{2\cos\theta}}\ .$$

The **total energy that must be supplied by the vehicle's energy system going up the hill** is the sum of the air drag energy loss and the increase in gravitational potential energy:

$$\boxed{E_u = E_{ud} + E_g = mCv^2\frac{l}{2\cos\theta} + mg\frac{l}{2}\tan\theta}\ ,$$

since the vehicle's kinetic energy remains constant going up the hill.

The **total energy change (increase) going down the hill** is

$$\boxed{E_d = E_g - E_{dd} = mg\frac{l}{2}\tan\theta - mCv^2\frac{l}{2\cos\theta}}\ .$$

That is, gravity provides energy and energy is lost due to air drag. Note that if $mg\frac{l}{2}\tan\theta > mCv^2\frac{l}{2\cos\theta}$, then **energy change will be positive going down the hill**; simplified, this condition is

$$\boxed{g\sin\theta > Cv^2}\ .$$

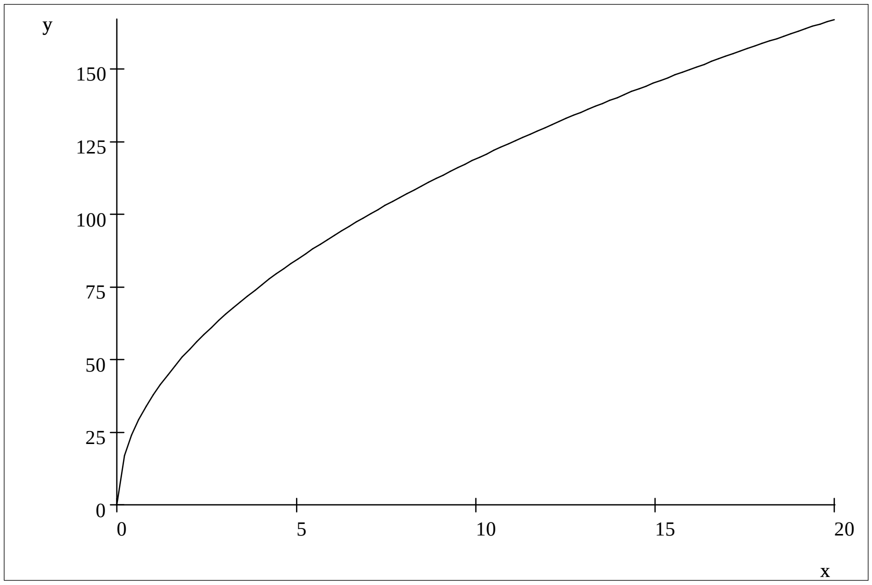
For the Toyota Prius II this yields:

- $\theta > 0.7\,\mathrm{deg}$ for a speed of $50\frac{\mathrm{km}}{\mathrm{hr}} = 31\frac{\mathrm{mi}}{\mathrm{hr}}$,
- $\theta > 2.7\,\mathrm{deg}$ for a speed of $100\frac{\mathrm{km}}{\mathrm{hr}} = 62\frac{\mathrm{mi}}{\mathrm{hr}}$,
- $\theta > 7.0\,\mathrm{deg}$ for a speed of $161\frac{\mathrm{km}}{\mathrm{hr}} = 100\frac{\mathrm{mi}}{\mathrm{hr}}$.

In terms of speed:

- $v < 86\frac{\mathrm{km}}{\mathrm{hr}} = 53\frac{\mathrm{mi}}{\mathrm{hr}}$ for a hill of angle $\theta = 2\,\mathrm{deg}$ hill,
- $v < 135\frac{\mathrm{km}}{\mathrm{hr}} = 84\frac{\mathrm{mi}}{\mathrm{hr}}$ for a hill of angle $\theta = 5\,\mathrm{deg}$ hill,
- $v < 192\frac{\mathrm{km}}{\mathrm{hr}} = 119\frac{\mathrm{mi}}{\mathrm{hr}}$ for a hill of angle $\theta = 10\,\mathrm{deg}$ hill.

In graph form:



Max. speed (y in mi/hr) for battery charging down a hill of angle

Note that the equation yields v in $\frac{\text{meter}}{\text{s}}$. To get it in $\frac{\text{mi}}{\text{hr}}$ multiply by 2.237 (http://www.onlineconversion.com/speed_all.htm).

So, some energy is available to charge the system's battery going down the hill for hills of large enough angle depending on the speed. This stored energy could then be used to help move the vehicle over the next part of the trip. If the hill angle $\theta < \arcsin(Cv^2/g)$, then the vehicle cannot travel down the hill at constant speed v; it will slow down as it goes down the hill.

The gasoline engine is turned off and the battery does not supply any electrical energy when going down the hill. So, the total energy that must be supplied by the vehicle's energy system to traverse the entire hill is just the energy required to travel up the hill, which is

$$\boxed{E = E_u = mCv^2 \frac{l}{2\cos\theta} + mg\frac{l}{2}\tan\theta}.$$

Since the vehicle is a hybrid, when going down the hill some of the gravitational potential can be converted to electrical energy by a generator to charge the vehicle's battery. The energy put into the battery is

$$E_s = fE_d = f(mg\frac{l}{2}\tan\theta - mCv^2\frac{l}{2\cos\theta})$$

where $f =$ the fraction $(0 < f < 1)$ of the available energy that is stored. The remaining available energy, $E_r = (1-f)E_d$, must be dissipated by some internal vehicle friction, mostly by friction or engine (non-firing) braking, to keep the speed constant when going down the hill.

The energy-per-distance (edt) traveled is

$$\text{edt} = \frac{mCv^2\frac{l}{2\cos\theta} + mg\frac{l}{2}\tan\theta}{2d} = \frac{1}{2}mCv^2 + \frac{1}{2}mg\cos\theta\tan\theta \quad \text{or}$$

$$\boxed{\text{edt} = \frac{1}{2}m(Cv^2 + g\sin\theta)}$$

The edt for gasoline only will be smaller, because some of the energy might be supplied by the battery. This is related to what the energy-usage display of the hybrid vehicle shows.

The energy-per-distance traveled going over the hill relative to the straight line between the beginning and ending points of the hill (edtf) (on the flat) is

$$\text{edtf} = \frac{mCv^2\frac{l}{2\cos\theta} + mg\frac{l}{2}\tan\theta}{l} = \frac{1}{2\cos\theta}mCv^2 + \frac{1}{2}mg\tan\theta \text{ or}$$

$$\boxed{\text{edtf} = \frac{1}{2}m(\frac{Cv^2}{\cos\theta} + g\tan\theta)}$$

This is probably the better quantity to compare to traveling on the flat, if the goal is to travel the horizontal distance l.

Traveling on the Flat

For traveling on the flat from one side of the triangle to the other, a distance l, the energy that must be supplied by the vehicle's energy system is $mCv^2 l$.

Then the energy-per-distance traveled is $\boxed{\text{edt} = mCv^2}$.

Comparison of Hill to Flat Travel

The ratio of energy-per-distance traveled, edt, for hill to flat travel without electric assistance is:

$$\boxed{R_{\text{edt}} = \frac{\frac{1}{2}m(Cv^2+g\sin\theta)}{mCv^2} = \frac{1}{2} + \frac{g\sin\theta}{2Cv^2}}$$

So, if $\frac{g\sin\theta}{2Cv^2} < \frac{1}{2}$ or $g\sin\theta < Cv^2$ or $\boxed{v > \sqrt{\frac{g\sin\theta}{C}}}$, then edt will be less for traveling over the hill than traveling on the flat.

- For the Prius II and a hill with $\theta = 2\,\text{deg}$: $v > 53\frac{\text{mi}}{\text{hr}}$. This is in the range of possibility for the Prius II.
- For the Prius II and a hill with $\theta = 5\,\text{deg}$: $v > 84\frac{\text{mi}}{\text{hr}}$. This is in the range of possibility for the Prius II.
- For the Prius II and a hill with $\theta = 10\,\text{deg}$: $v > 119\frac{\text{mi}}{\text{hr}}$, well beyond the Prius II's maximum speed of $103\frac{\text{mi}}{\text{hr}}$.

Note that the maximum ratio is $R_{\text{edt}}(v \to \infty) = \frac{1}{2}$.

One could argue that the comparsion should be made between edtf instead of edt. Then the ratio is:

$$\boxed{R_{\text{edtf}} = \frac{\frac{1}{2}m(\frac{Cv^2}{\cos\theta}+g\tan\theta)}{mCv^2} = \frac{Cv^2+g\sin\theta}{2Cv^2\cos\theta}}.$$

For this ratio to be less than one: $Cv^2 + g\sin\theta < 2Cv^2\cos\theta$ or $g\sin\theta < Cv^2(2\cos\theta - 1)$. Therefore, $\boxed{v > \sqrt{\frac{g\sin\theta}{C(2\cos\theta-1)}}}$.

- For the Prius II on a hill with $\theta = 2\,\text{deg}$: $v > 53\frac{\text{mi}}{\text{hr}}$. This is in the range of possibility for the Prius II.

- For the Prius II on a hill with $\theta = 5\,\text{deg}$: $v > 85\frac{\text{mi}}{\text{hr}}$. This is in the range of possibility for the Prius II.
- For the Prius II on a hill with $\theta = 10\,\text{deg}$: $v > 121\frac{\text{mi}}{\text{hr}}$, which is well beyond the Prius II's maximum speed of $103\frac{\text{mi}}{\text{hr}}$.

Note that the maximum ratio is $R_{\text{edt}}(v \to \infty) = \frac{1}{2\cos\theta}$ for hill angle θ.

Remember that these calculations involve the total system energy used to move the hybrid vehicle up the hill. Some of that energy is usually supplied by the battery. It does not account for the energy stored in the battery going
down the hill.

Case of Air Drag, Brake Regeneration and Electric Assistance Going up the Hill

Traveling Over the Hill

Assume that the energy recovered by charging the battery when going down the hill is the same as the energy supplied by the battery going up the hill. Then, the **energy supplied by the gasoline engine** going up the hill is

$$E_u = mCv^2\frac{l}{2\cos\theta} + mg\frac{l}{2}\tan\theta - f(mg\frac{l}{2}\tan\theta - mCv^2\frac{l}{2\cos\theta}) \text{ or}$$

$$\boxed{E_u = (1+f)mCv^2\frac{l}{2\cos\theta} + (1-f)mg\frac{l}{2}\tan\theta},$$

where $f =$ the fraction of available energy that is stored in the battery when going down the hill $(0 < f < 1)$.

The gasoline-energy-per-distance (edt) traveled is

$$\text{edt(gasoline)} = \frac{(1+f)mCv^2\frac{l}{2\cos\theta}+(1-f)mg\frac{l}{2}\tan\theta}{2d} = (1+f)\frac{1}{2}mCv^2 + (1-f)\frac{1}{2}mg\cos\theta\tan\theta \text{ or}$$

$$\boxed{\text{edt} = \frac{1}{2}m[(1+f)Cv^2 + (1-f)g\sin\theta]}$$

This is related to what the gasoline usage display of the vehicle shows.

The gasoline-energy-per-distance traveled going up the hill relative to the straight line between the beginning and ending points of the hill (edtf) (on the flat) is

$$\boxed{\text{edtf(gasoline)} = \frac{(1+f)mCv^2\frac{l}{2\cos\theta}+(1-f)mg\frac{l}{2}\tan\theta}{l} = \frac{1}{2}m[(1+f)Cv^2\frac{1}{\cos\theta} + (1-f)g\tan\theta]}$$

This is probably the better quantity to compare to traveling on the flat, if the goal is to travel the horizontal distance l.

Traveling on the Flat

It is probably more accurate to assume that 10% of the energy needed to travel on the flat is supplied by the battery when the vehicle is traveling on the flat, with the rest supplied by gasoline.

Then the gasoline-energy-per-distance traveled supplied by is $\boxed{\text{edt(gasoline)}=0.9mCv^2}$.

Comparison of Hill to Flat Travel

The ratio of gasoline-energy-per-distance traveled, edt, for hill to flat travel with 10% electric assistance for flat travel is:

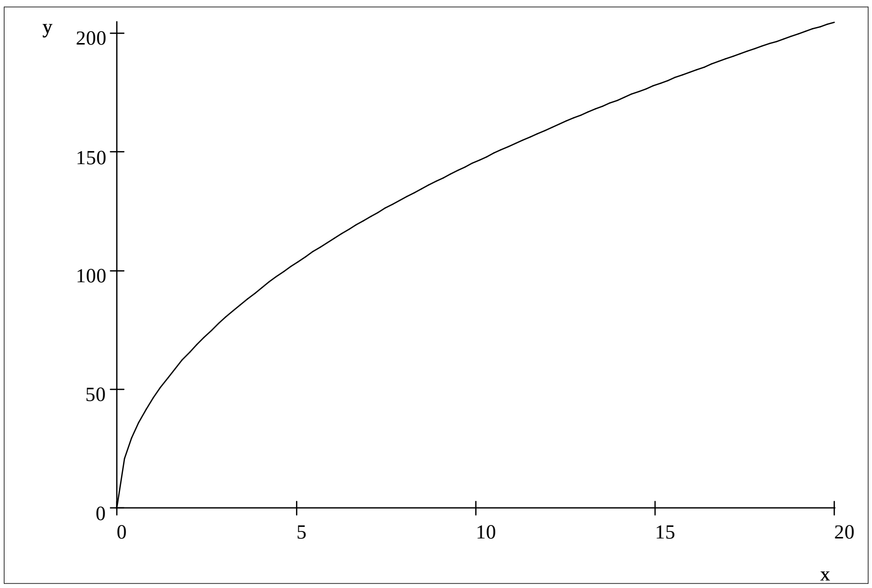
$$\boxed{R_{\text{edt}}(\text{gasoline}) = \frac{\frac{1}{2}m[(1+f)Cv^2+(1-f)g\sin\theta]}{0.9mCv^2} = \frac{(1+f)Cv^2+(1-f)g\sin\theta}{0.9(2Cv^2)}}.$$

For this ratio to be less than one: $(1+f)Cv^2 + (1-f)g\sin\theta < 1.8Cv^2$ or $(1-f)g\sin\theta < (0.8-f)Cv^2$ or

$$\boxed{v > \sqrt{\frac{(1-f)g\sin\theta}{(0.8-f)C}}}.$$

- For the Prius II on a hill of $2\,\text{deg}$ and $f = 0.4$: $v > 65\frac{\text{mi}}{\text{hr}}$. This is in the range of possibility for the Prius II.
- For the Prius II on a hill of $5\,\text{deg}$ and $f = 0.4$: $v > 103\frac{\text{mi}}{\text{hr}}$. This is at the end of the range of possibility for the Prius II.
- For the Prius II on a hill of $10\,\text{deg}$ and $f = 0.4$: $v > 146\frac{\text{mi}}{\text{hr}}$. This is well beyond the range of possibility for the Prius II.

In graph form:



Min. speed (mi/hr) for less energy going over hill of angle x

Note that the equation yields v in $\frac{\text{meter}}{\text{s}}$. To get it in $\frac{\text{mi}}{\text{hr}}$ multiply by 2.237 (http://www.onlineconversion.com/speed_all.htm).

For 3 different values of $f\,(0.2, 0.4, 0.6)$ and 3 different values of $\theta(2\,\text{deg}, 5\,\text{deg}, 10\,\text{deg})$ the minimum speeds $\left(\frac{\text{mi}}{\text{hr}}\right)$ are:

f/θ	2	5	10
0.2	62	97	137
0.4	65	103	146
0.6	76	119	168

One could argue that the comparsion should be made between edtf instead of edt. Then the ratio is:

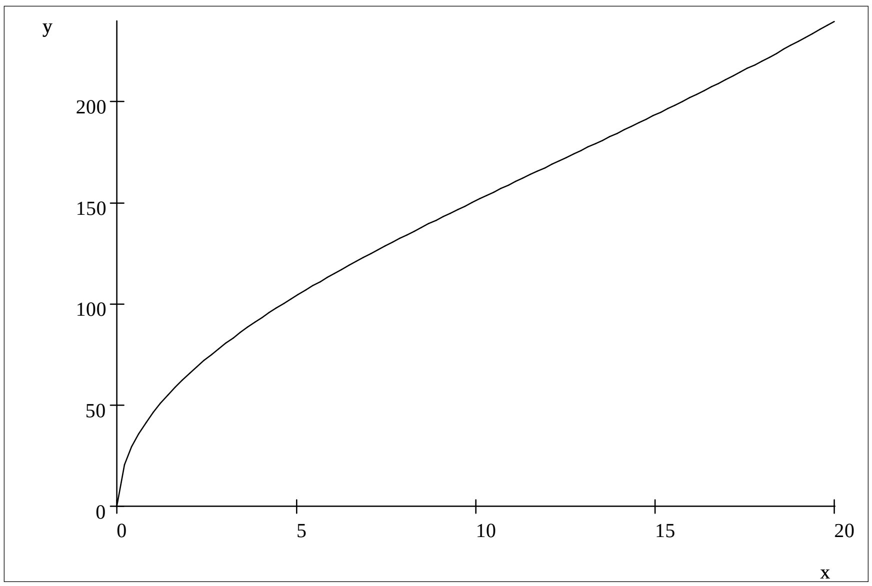
$$\boxed{R_{\text{edtf}}(\text{gasoline}) = \frac{\frac{1}{2}m\left[(1+f)Cv^2\frac{1}{\cos\theta}+(1-f)g\tan\theta\right]}{0.9mCv^2} = \frac{(1+f)Cv^2+(1-f)g\sin\theta}{0.9(2Cv^2\cos\theta)}}.$$

For this ratio to be less than one: $(1+f)Cv^2 + (1-f)g\sin\theta < 1.8Cv^2\cos\theta$ or $(1-f)g\sin\theta < Cv^2(1.8\cos\theta - 1 - f)$ or

$$\boxed{v > \sqrt{\frac{(1-f)g\sin\theta}{C(1.8\cos\theta-1-f)}}}.$$

- For the Prius II on a hill with $\theta = 2\,\text{deg}$ and $f = 0.4$: $v > 66\frac{\text{mi}}{\text{hr}}$. This is in the range of possibility for the Prius II.
- For the Prius II on a hill with $\theta = 5\,\text{deg}$ and $f = 0.4$: $v > 104\frac{\text{mi}}{\text{hr}}$. This is at the end of the range of possibility for the Prius II.
- For the Prius II on a hill with $\theta = 10\,\text{deg}$ and $f = 0.4$: $v > 151\frac{\text{mi}}{\text{hr}}$, well beyond the Prius' maximum speed of $103\frac{\text{mi}}{\text{hr}}$.

In graph form:



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Note that the equation yields v in $\frac{\text{meter}}{\text{s}}$. To get it in $\frac{\text{mi}}{\text{hr}}$ multiply by 2.237 (http://www.onlineconversion.com/speed_all.htm).

For 3 different values of $f\,(0.2, 0.4, 0.6)$ and 3 different values of $\theta(2\,\text{deg}, 5\,\text{deg}, 10\,\text{deg})$ the minimum speeds $\left(\frac{\text{mi}}{\text{hr}}\right)$ are:

f/θ	2	5	10
0.2	62	98	136
0.4	66	104	141
0.6	76	121	151

Conclusion

So, for the assumptions given in this paper, a hybrid vehicle can use less gasoline going over a hill than it would use traveling on the flat for the same horizontal distance if the speed is greater than a certain value, depending on the angle of the hill.

For a hill of $2\,\mathrm{deg}$ angle with battery energy used to supplement gasoline going up the hill equal to 40% of the downhill energy change and 10% of the system energy on the flat provided by the battery, the energy display would show less gasoline used going over the hill compared to traveling on the flat if the speed is $v > 65\,\frac{\mathrm{mi}}{\mathrm{hr}}$.

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