Rationalizing the Hodgkin-Huxley Equations for Nerve Excitation

L. David Roper, roperld@vt.edu, 2 June 2024
http://roperld.com/personal/RoperLDavid.htm

Introduction

Hodgkin and Huxley (HH) (https://en.wikipedia.org/wiki/Hodgkin%E2%80%93Huxley_model) introduced some mathematical equations to describe nerve excitation in 1954..

Total current flow:

$$I_m = C_m dV/dt + I_{Na} + I_K + I_{leak}$$

Ohms Law applied:

$$I_{Na} = (V - V_{Na})g_{Na} = (V - V_{Na})\overline{g}_{Na} m^3 h$$
$$I_K = (V - V_K)g_K = (V - V_K)\overline{g}_K n^4$$
$$I_{leak} = (V - V_{leak})g_{leak}$$

Na and K channels opening, closing (and inactivating):

$$dm/dt = \alpha_m(1-m) - \beta_m m$$
$$dh/dt = \alpha_h(1-h) - \beta_h h$$
$$dn/dt = \alpha_n(1-n) - \beta_n n$$

The rates at which the channels open and close are voltage dependent:

$$\alpha_m = 0.1\frac{V+25}{\exp\left(\frac{V+25}{10}\right)-1} \qquad \beta_m = 4\exp\left(\frac{V}{18}\right)$$
$$\alpha_h = 0.07\exp\left(\frac{V}{20}\right) \quad \beta_h = \frac{1}{\exp\left(\frac{V+30}{10}\right)+1}$$
$$\alpha_n = 0.01\frac{V+10}{\exp\left(\frac{V+10}{10}\right)-1} \qquad \beta_n = 0.125\exp\left(\frac{V}{80}\right)$$

Here V = depolarization potential inward.

HH derived these equations by fitting voltage-clamp data for mV $-56 \le V_c \le +49$ in milliVolts. The resting potential is -62 mV, so the depolarization-potential range is $+6 \le V \le +109$.

Voltage Dependence of Na and K Channels

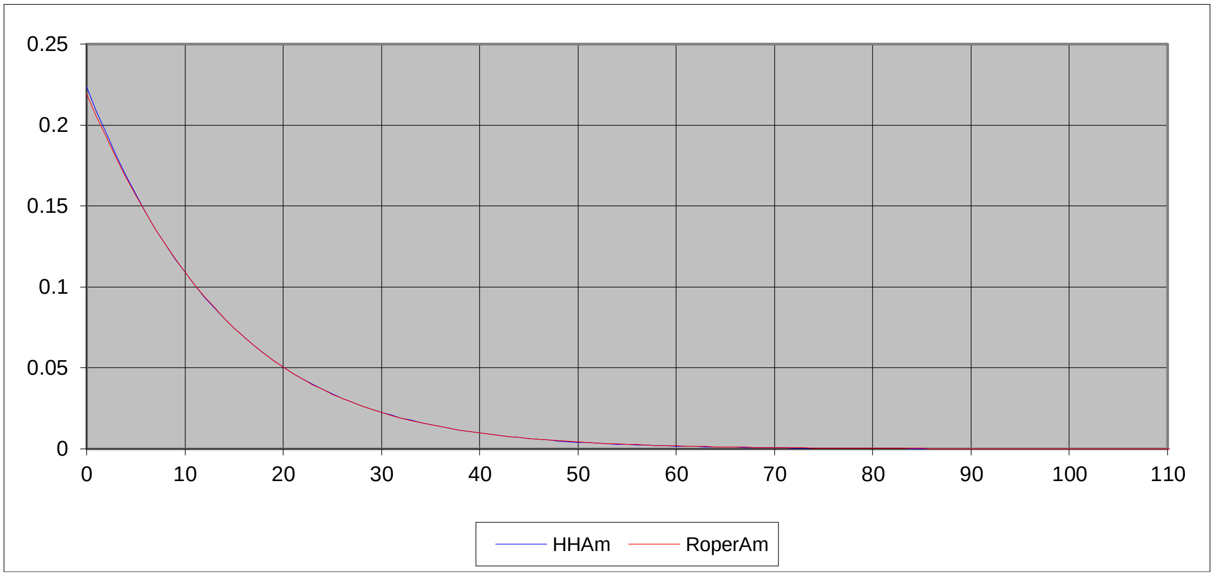
Consider the last set of equations that describe the voltage dependence of the Na and K channels. Functions that can achieve infinite values are generally not appropriate functions to represent physical data such as nerve excitation data. Of the six equations given directly above, only one behaves properly for large V: $\beta_h = \frac{1}{\exp\left(\frac{V+30}{10}\right)+1}$ ($\to 0$ for $V \to -\infty$ and $\to 1$ for $V \to \infty$). Two of the equations become infinitely large linear in V: $\alpha_m = 0.1\frac{V+25}{\exp\left(\frac{V+25}{10}\right)-1}$ and $\alpha_n = 0.01\frac{V+10}{\exp\left(\frac{V+10}{10}\right)-1}$. Three of the equations become infinitely large exponentially in V:

$\beta_m = 4\exp\left(\frac{V}{18}\right)$ for positive V, $\alpha_h = 0.07\exp\left(\frac{V}{20}\right)$ for negative V and $\beta_n = 0.125\exp\left(\frac{V}{80}\right)$ for positive V.

What is needed are functions that move from one asymptote to another asymptote as the voltage changes. The two asymptotes represent two stable states of the nerve cell. The hyperbolic tangent (tanh) is such a function (http://www.roperld.com/science/HyperbolicTangentWorld.pdf).

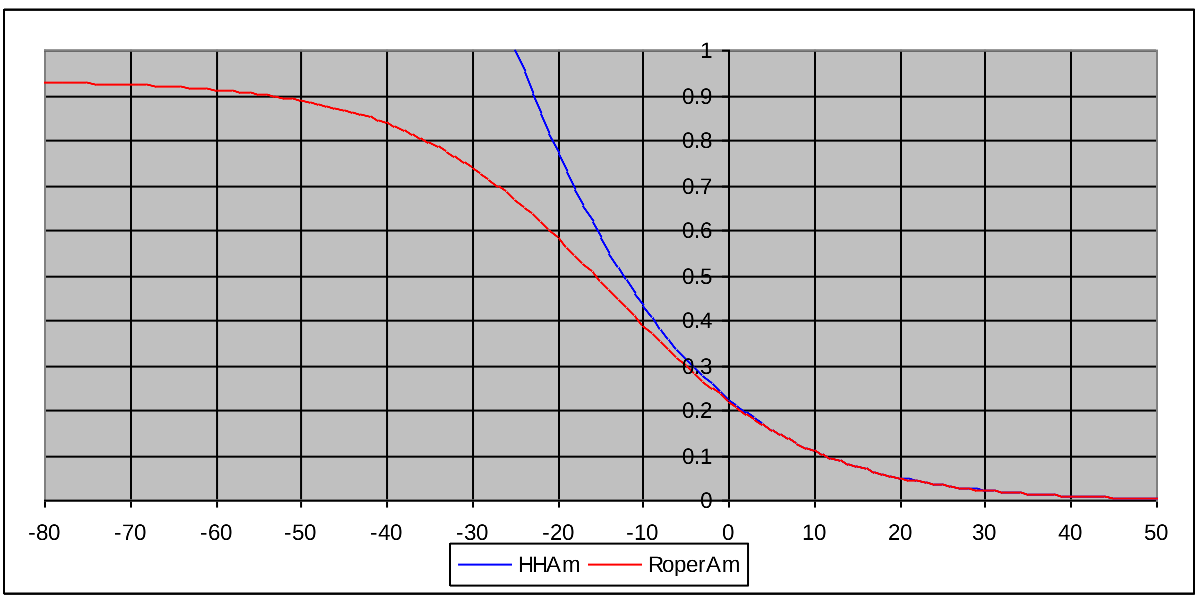
So, I have fitted tanh functions to the values as calculated by the HH voltage-dependent equations over the range +6 mV <= V <= +109 mV, where V = depolarization voltage. The result are shown in the graphs and equations below.

The α_m equation:

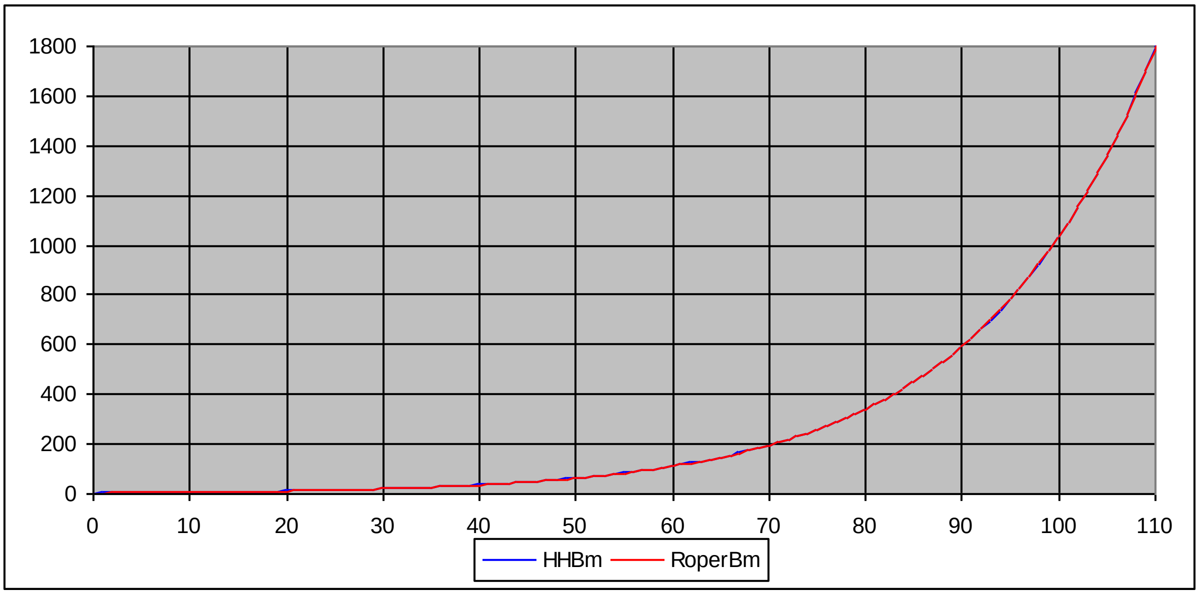


α_m = 0.465(1-tanh((V+14.0)/23.8).

Note the minus sign before the tanh function, which is required to move from a higher state to a lower state. transition point is negative [-(-14) = +14)]. This curve does not show the asymptotic behavior; so the following graph shows it for a stretched-out ordinate:



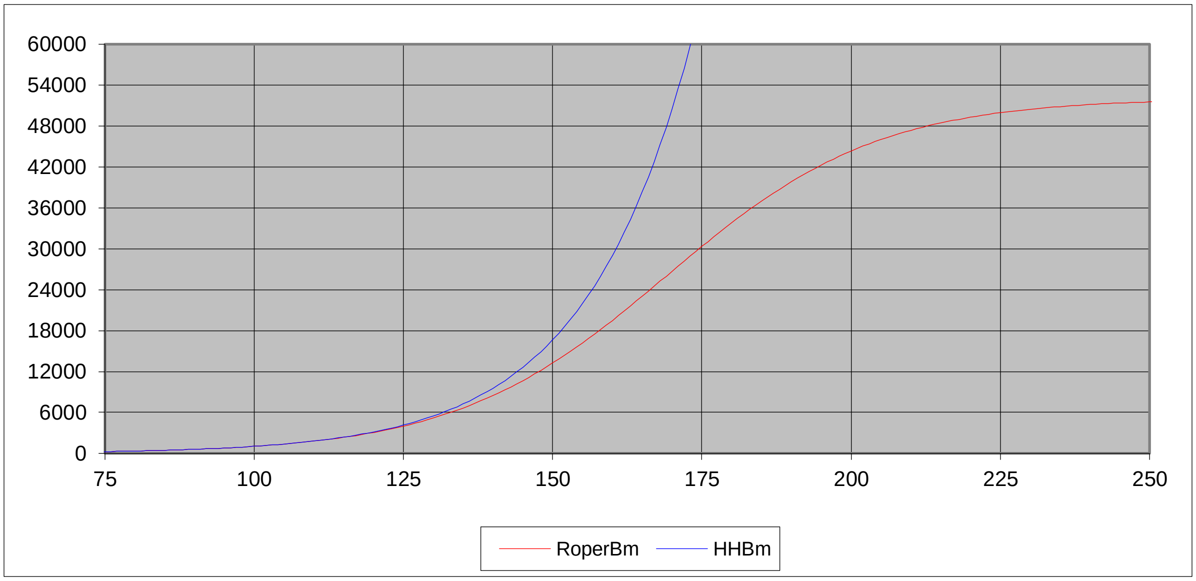
The β_m equation:



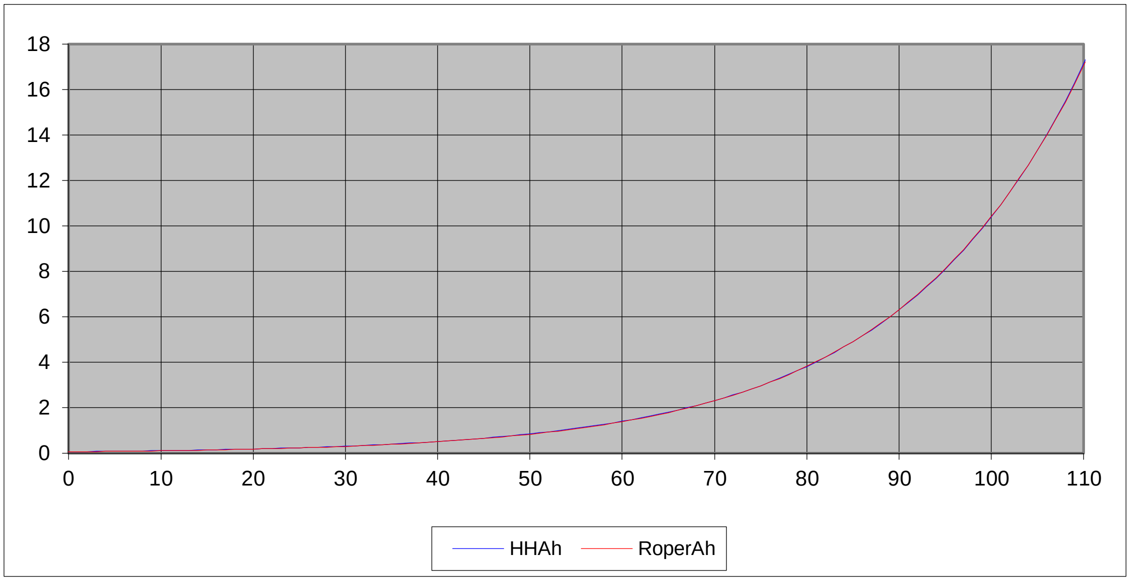
$$\beta_m = \mathbf{26000(1+tanh((V\text{-}169)/35.5)}.$$

The fit is so close that the two curves are not visually separated.

This curve does not show the asymptotic behavior; so the following graph shows it for a stretched-out ordinate:



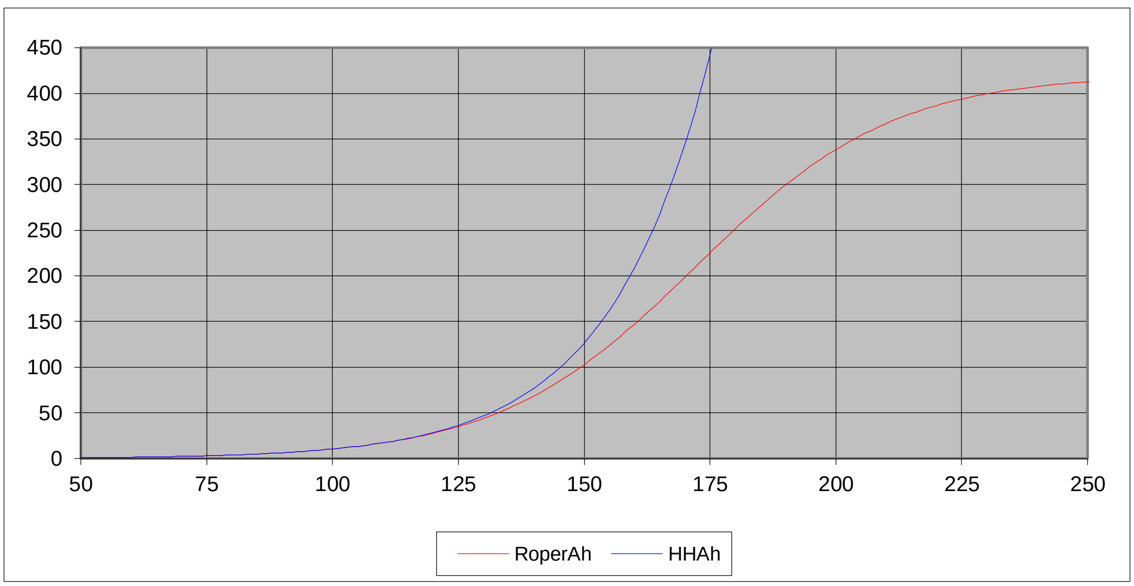
The α_h equation:



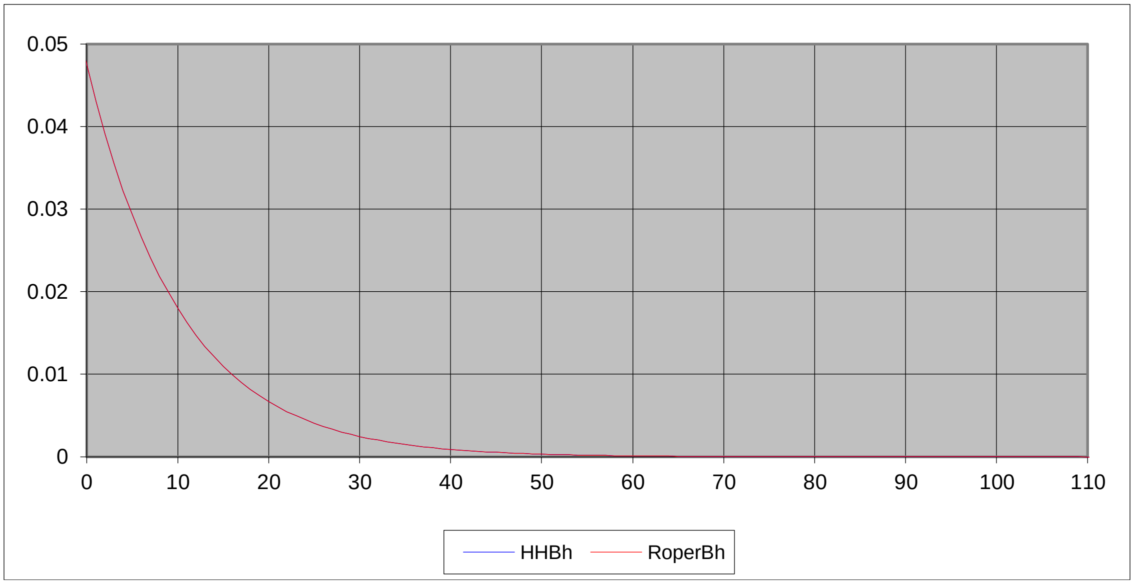
$$\alpha_h = 210(1+\tanh((V-172)/39.3)).$$

The fit is so close that the two curves are not visually separated.

This curve does not show the asymptotic behavior; so the following graph shows it for a stretched-out ordinate:



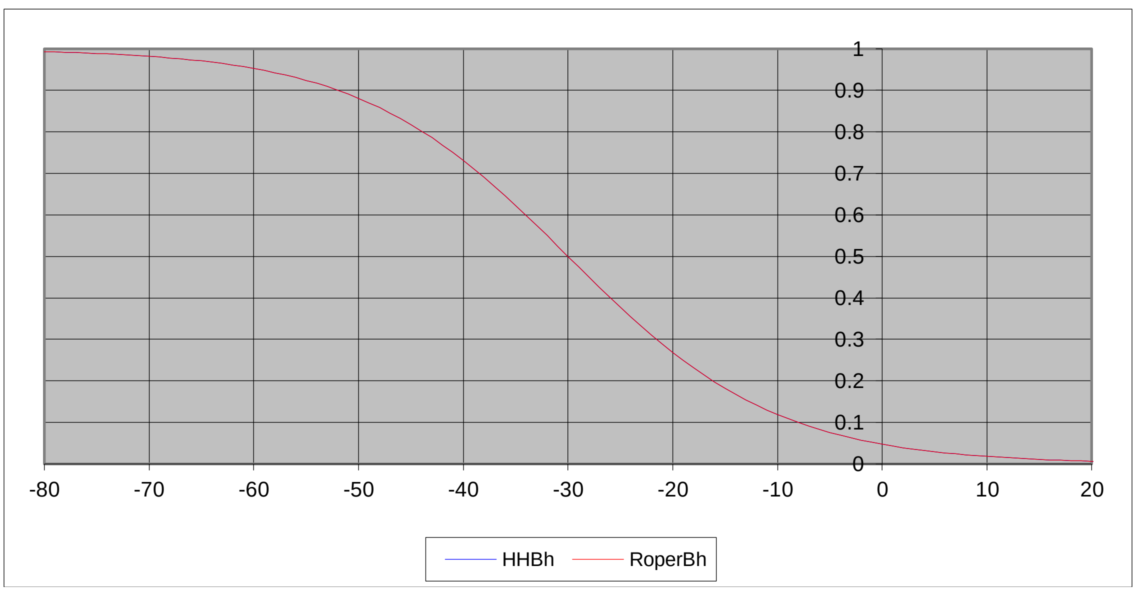
The β_h equation:



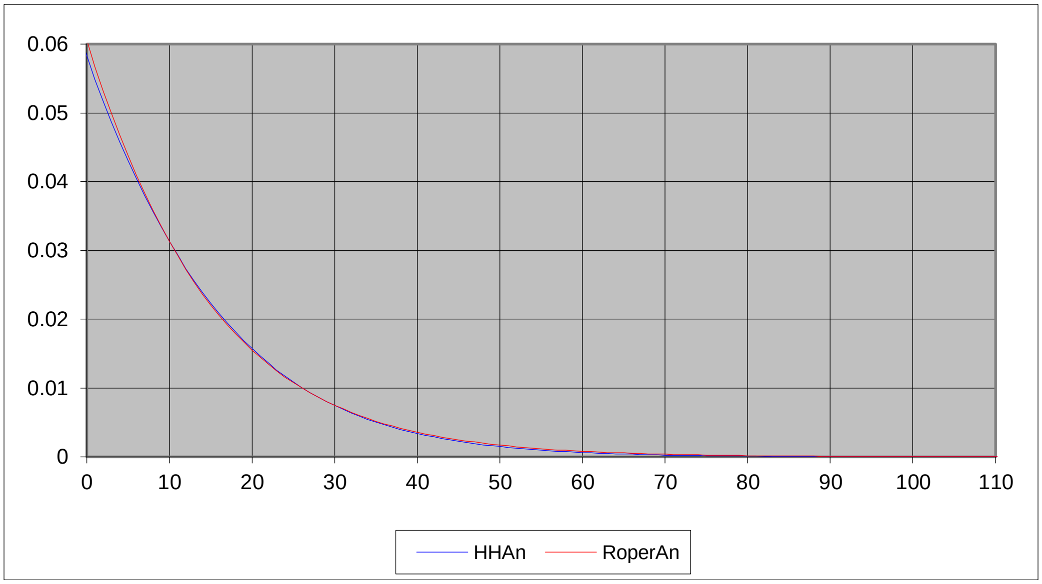
β_h **= 0.500(1-tanh((V+30.0)/20.0)**.

Note the minus sign before the tanh function, which is required to move from a higher state to a lower state. The transition point is negative [-(-30) = +30)]. The fit is so close that the two curves are not visually separated.

This curve does not show the asymptotic behavior; so the following graph shows it for a stretched-out ordinate:

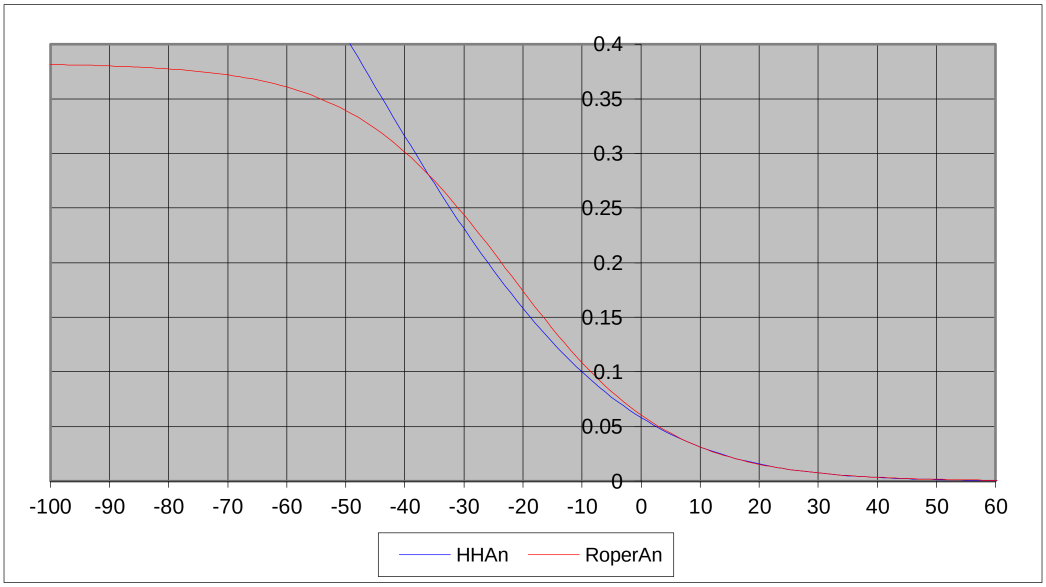


The α_n equation:

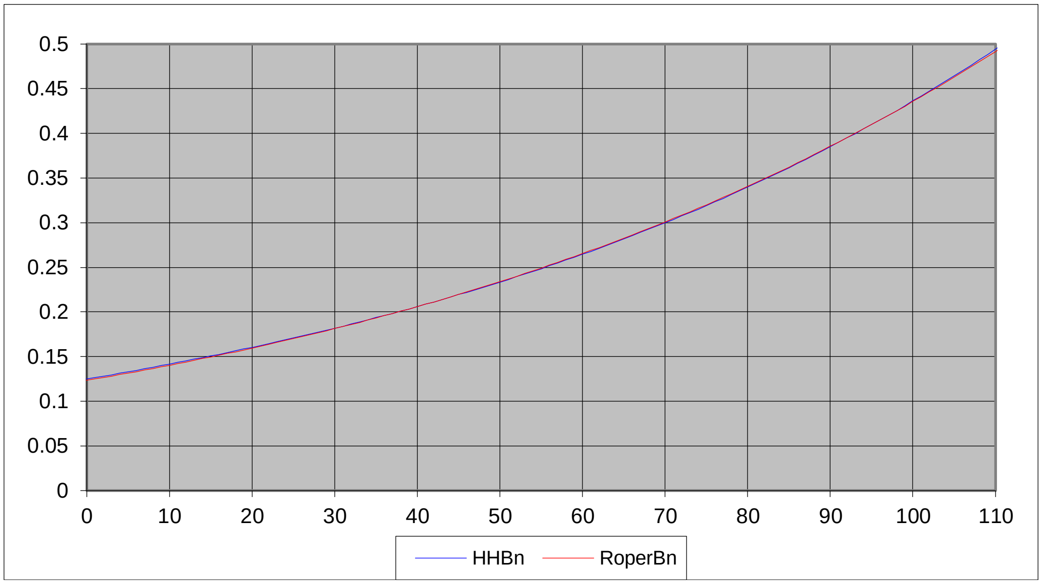


α_n = 0.191(1-tanh((V+22.4)/26.8).

Note the minus sign before the tanh function, which is required to move from a higher state to a lower state. transition point is negative [-(-22.4) = +22.4)]. This curve does not show the asymptotic behavior; so the following graph shows it for a stretched-out ordinate:

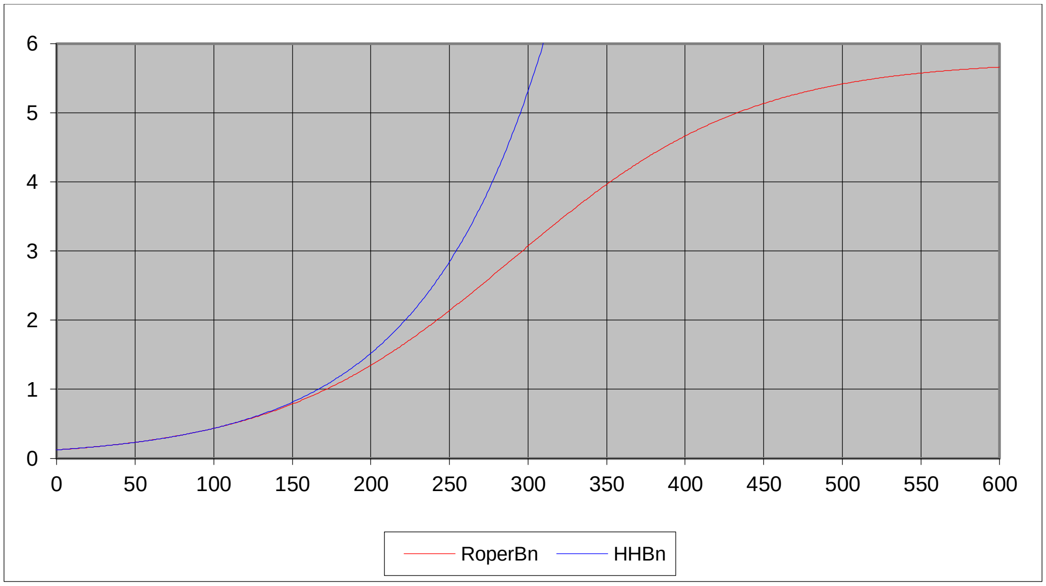


The β_n equation:



β_n **= 2.88(1+tanh((V-290)/152).**

This curve does not show the asymptotic behavior; so the following graph shows it for a stretched-out ordinate:



Conclusion

The set of rationalized equations is:

$$\alpha_m = \mathbf{0.465(1-tanh((V+14.0)/23.8)}$$
$$\beta_m = \mathbf{26000(1+tanh((V-169)/35.5)}$$
$$\alpha_h = \mathbf{210(1+tanh((V-172)/39.3)}$$
$$\beta_h = \mathbf{0.500(1-tanh((V+30.0)/20.0)}$$
$$\alpha_n = \mathbf{0.191(1-tanh((V+22.4)/26.8)}$$
$$\beta_n = \mathbf{2.88(1+tanh((V-290)/152)}$$

There may be other sets of parameters that yield equally good fits to the HH values in +6 mV <= V <= +109 mV range. This work just shows that it is possible to fit the HH values with physically reasonable equations that asymptotically remain finite.

Replacing the HH equations with these equations yields a much more physically reasonable theory regarding the channel states and the transitions between them.