Scattering Partial-Wave Equations and Resonance Equations

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Preface to August 2010 Revision

1. I have made a few corrections and added a few equations.

2. I have put the figures in a separate file, UCRL14193_RoperLD_Figures.pdf, so that it can be viewed beside this document. The captions for the figures are included here as well as in the figures file.

3. In some cases the figures did not scan very clearly from the original document.

Abstract

This report contains equations that relate the phase shifts and partial-wave amplitudes for elastic scattering. The effect of inelasticity on elastic scattering is included. Also, resonance equations are given, and many curves show the behavior of a resonance as a function of the various resonance parameters for the case of $\pi - p$ scattering. In particular, the unusual behavior of highly absorptive resonances is emphasized. (For the latest results for $\pi - p$ scattering see http://gwdac.phys.gwu.edu/analysis/pin_analysis.html .)

Contents

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I. Introduction

Herein are listed many familiar equations regarding partial-wave scattering amplitudes and phase shifts. The unusual behavior of inelastic resonances is elucidated algebraically and graphically. We shall always consider the case where absorption occurs, as the elastic case can be obtained by setting $\eta = 1$.

We define:

$S = \eta e^{2i\delta}$	S matrix element for elastic scattering (unitarity requires that $S^{\dagger}S = 1$)
$A = \frac{S-1}{2i} = \frac{1}{2i}(\eta e^{2i\delta} - 1)$	Partial-wave amplitude for elastic scattering
δ	Phase shift for elastic scattering
$\eta = e^{-2\nu}$	Absorption parameter (a measure of the incident particles removed from the beam due to inelastic scattering) (ν is the imaginary part of the phase shift. We shall always use η rather than ν .)

Subscripts denoting the angular momentum, parity, and isotopic spin have been suppressed on the quantities above.

$k =$	c.m. momentum (in units of incident particle mass)
$q_0 =$	Incident particle c.m. total energy (in units of incident particle mass)
$E =$	Incident particle lab. kinetic energy (MeV)

The equations relating the last three quantities are

$$k = \frac{M_T}{M_i}\sqrt{\frac{E(E+2M_i)}{(M_T+M_i)^2 + 2M_TE}} \text{ and } q_0 = \sqrt{k^2+1} \ ,$$

where M_T and M_i are the target particle and incident particle masses (MeV), respectively. The total c.m. energy is $W = p_0 + q_0$, where the target particle c.m. total energy is

$$p_0 = \sqrt{k^2 + \left(\frac{M_T}{M_i}\right)^2} \ ,$$

both in units of the incident particle mass. (The speed of light c is set to 1.)

More kinematics equations are at the end.

II. Partial-Wave Amplitude Equations

The partial-wave amplitude is

$$A = \frac{1}{2}(\eta e^{2i\delta} - 1) = \eta e^{i\delta}\sin\delta + \frac{i}{2}(1-\eta)\left\{\begin{array}{l} \xrightarrow[\eta=1]{} e^{i\delta}\sin\delta \\ \hline \xrightarrow[\eta=0]{} \frac{i}{2} \end{array}\right| \qquad 1$$

(See Fig. 1 for loci of constant η and δ . See Ref. 1 for relationship between partial-wave amplitudes and observables.)

Therefore:

$$\mathrm{Re}A = \frac{1}{2}\eta\sin 2\delta \left\{ \begin{array}{c} \underset{\eta=1}{\longrightarrow} \ \sin\delta\cos\delta \\ \hline \underset{\eta=0}{\longrightarrow} \ 0 \end{array} \right| , \ \mathrm{Im}A = \frac{1}{2}(1-\eta\cos 2\delta) \left\{ \begin{array}{c} \underset{\eta=1}{\longrightarrow} \ \sin^2\delta \\ \hline \underset{\eta=0}{\longrightarrow} \ \frac{1}{2} \end{array} \right| \tag{2}$$

$$\left(-\frac{1}{2} \le \mathrm{Re}A \le \frac{1}{2}\right) \qquad\qquad (0 \le \mathrm{Im}A \le 1) \tag{\#}$$

$$\tan 2\delta = \frac{2\mathrm{Re}A}{1-2\mathrm{Im}A} \ , \ \eta^2 = (2\mathrm{Re}A)^2 + (1-2\mathrm{Im}A)^2 \tag{3}$$

$$\tan\delta = \frac{2\mathrm{Im}A - 1 + \sqrt{(2\mathrm{Re}A)^2 + (1-2\mathrm{Im}A)^2}}{2\mathrm{Re}A} = \frac{2\mathrm{Im}A - 1 + \eta}{2\mathrm{Re}A} \underset{\eta=1}{\longrightarrow} \frac{\mathrm{Im}A}{\mathrm{Re}A} \tag{4}$$

Fig. 1. Loci of partial-wave amplitudes in the complex plane for constant η [circles of radii $\eta/2$ centered at (0,1/2)] and for constant δ [radial lines emanating from (0,1/2) to a distance of 1/2]. An arbitrary amplitude must lie inside or on the outer circle.

Other ways to write the relationships listed above:

$$S = \eta e^{2i\delta} = \eta\frac{1+i\tan\delta}{1-i\tan\delta} \ . \tag{5}$$

$$A = \frac{S-1}{2i} = \frac{i(1-\eta)+(\eta+1)\tan\delta}{2(1-i\tan\delta)} \underset{\eta=1}{\longrightarrow} \frac{\tan\delta}{1-i\tan\delta} = \frac{1}{\cot\delta - i} \ ; \tag{6}$$

therefore,

$$\frac{1}{A} \underset{\eta=1}{\longrightarrow} \cot\delta - i \ ; \tag{7}$$

$$\mathrm{Im}A = |A|^2 + \frac{1}{4}(1-\eta^2) \underset{\eta=1}{\longrightarrow} |A|^2 \tag{8}$$

III. Resonance Formulas

Define:

q_{0_r}	Resonance position incident particle total c.m. energy (in units of incident particle mass)
E_r	Resonance energy in terms of lab. kinetic energy of incident particle (MeV)
Γ_{el}	Elastic full width at half maximum (in units of incident particle mass)
Γ_{in}	Inelastic full width at half maximum (in units of incident particle mass)
$\Gamma = \Gamma_{el} + \Gamma_{in}$	Total full width at half maximum
$x = \frac{\Gamma_{el}}{\Gamma}$	Fractional elastic width
$\epsilon = \frac{2}{\Gamma}(q_{0_r} - q_0)$	Distance from the resonance position in units of the half width at half maximum

The Breit-Wigner resonance formula is[2,3]

$$A_R = \frac{\Gamma_{el}}{2(q_{0_r} - q_0) - i\Gamma} = \frac{x}{\epsilon - i} \left\{ \begin{array}{c} \underset{x=1}{\longrightarrow} \frac{1}{\epsilon - 1} \\ \hline \underset{x=0}{\longrightarrow} 0 \end{array} \right| . \qquad 9$$

(See Fig. 2) ($x = 1$ represents pure elastic scattering. $x = 0$ represents no elastic scattering at all).

$$S_R = 1 + 2iA_R = \frac{\epsilon - 1 + 2ix}{\epsilon - i} \left\{ \begin{array}{c} \underset{x=1}{\longrightarrow} \frac{\epsilon + i}{\epsilon - i} \\ \hline \underset{x=0}{\longrightarrow} 1 \end{array} \right| .$$

Fig. 2. Loci of resonance partial-wave amplitudes[1] in the complex plane for constant x (circles of radii $x/2$ centered at $(0, x/2)$ and for constant ϵ (radial lines emanating from (0.0) to the outer circle). In general x is a slowly varying function of ϵ , so that the actual loci approximate circles.

Therefore:

$$\mathrm{Re} A_R = \frac{\epsilon x}{\epsilon^2 + 1} \left\{ \begin{array}{c} \underset{x=1}{\longrightarrow} \frac{\epsilon}{\epsilon^2 + 1} \\ \hline \underset{x=0}{\longrightarrow} 0 \end{array} \right| , \ \mathrm{Im} A_R = \frac{x}{\epsilon^2 + 1} \left\{ \begin{array}{c} \underset{x=1}{\longrightarrow} \frac{1}{\epsilon^2 + 1} \\ \hline \underset{x=0}{\longrightarrow} 0 \end{array} \right| \qquad 11$$

and

$$\tan 2\delta_R = \frac{2\epsilon x}{\epsilon^2 + 1 - 2x} \left\{ \begin{array}{c} \underset{x=1}{\longrightarrow} \frac{2\epsilon}{\epsilon^2 - 1} \underset{\epsilon=0}{\longrightarrow} -2\epsilon \text{ (i.e., } \delta_R \underset{\epsilon \to 0}{\longrightarrow} \pi/2) \\ \hline \underset{x=0}{\longrightarrow} 0 \text{ (i.e., } \delta_R = 0 \text{ for all values of } \epsilon) \end{array} \right| , \qquad 12$$

$$\eta_R^2 = \frac{\epsilon^2 + (2x - 1)^2}{\epsilon^2 + 1} \left\{ \begin{array}{c} \underset{x=1}{\longrightarrow} 1 \\ \underset{x=1/2}{\longrightarrow} \frac{\epsilon^2}{\epsilon^2 + 1} \underset{\epsilon=0}{\longrightarrow} 0 \\ \underset{x=0}{\longrightarrow} 1 \end{array} \right| ; \qquad 13$$

or

$$\tan \delta_R = \frac{2x - \epsilon^2 - 1 + \sqrt{(\epsilon^2 + 1)\left[\epsilon^2 + (2x - 1)^2\right]}}{2\epsilon x} \underset{x=1}{\longrightarrow} \frac{1}{\epsilon} = \frac{\Gamma}{2(q_{0_r} - q_0)} \qquad 14$$

and

$$\tan \delta_R \underset{\epsilon \to 0}{\longrightarrow} \frac{2x - 1 + \sqrt{(2x - 1)^2}}{2\epsilon x} \left\{ \begin{array}{c} \underset{x > 1/2}{\longrightarrow} \infty \text{ (i.e., } \delta_R = \pi/2) \\ \hline \underset{x < 1/2}{\longrightarrow} 0 \text{ (i.e., } \delta_R = 0 \text{)} \end{array} \right|$$

Equations (11) through (14) exhibit the following behavior (see Fig. 5):

Considering x as constant with respect to ϵ (more about this later):

(a) for $x > 1/2$ the phase shift (δ_R) passes through 90° at $\epsilon = 0$ and asymptotically approches 180° as $\epsilon \to -\infty$; the absorption parameter (η_R) symmetrically dips (with respect to ϵ) to a minimum ($\eta_{R\min} = 2x - 1$) at $\epsilon = 0$ and asymptotically approaches 1 as $\epsilon \to -\infty$;

(b) for $x < 1/2$ the phase shift (δ_R) passes through 0° at $\epsilon = 0$ after having reached a maximum $[\delta_{R\max} = \frac{1}{2}\tan^{-1}\left(x/\sqrt{1-2x}\right) < 45^\circ]$ at $\epsilon = \sqrt{1-2x}$. It reaches a minimum $[\delta_{R\min} = \frac{1}{2}\tan^{-1}\left(-x/\sqrt{1-2x}\right) > -45^\circ]$ at $\epsilon = -\sqrt{1-2x}$ and asymptotically approaches 0° at $\epsilon \to -\infty$; the absorption parameter symmetrically dips to a minimum $(\eta_{R\min} = 1-2x)$ at $\epsilon = 0$ and asymptotically approaches 1 as $\epsilon \to -\infty$.

(c) for for $x = 1/2$ the phase shift (δ_R) at $\epsilon = 0$ as a function of x is discontinuous at $x = 1/2$ $[\delta_R(\epsilon = 0, x = 1/2 + \Delta) = 90^\circ, \delta_R(\epsilon = 0, x = 1/2 - \Delta) = 0^\circ$ where Δ is a small positive number, as shown below]; the absorption parameter symmetrically dips to zero at $\epsilon = 0$ and asymptotically approaches 1 as $\epsilon \to -\infty$.

$$\tan 2\delta_R \xrightarrow[x = 1/2+\Delta]{} \frac{\epsilon(1+2\Delta)}{\epsilon^2 - 2\Delta} \left\{ \begin{array}{l} \underset{\epsilon\to 0}{\longrightarrow} \frac{\epsilon(1+2\Delta)}{-2\Delta} \underset{\Delta\to 0}{\longrightarrow} -0 \text{ (i.e., } \delta_R = 90^\circ) \\ \underset{\Delta\to 0}{\longrightarrow} \frac{1}{\epsilon} \underset{\epsilon\overset{\to}{\pm}0}{\longrightarrow} \pm\infty \text{ (i.e., } \delta_R = \pm 45^\circ) \end{array} \right|$$

and

$$\tan 2\delta_R \xrightarrow[x = 1/2-\Delta]{} \frac{\epsilon(1-2\Delta)}{\epsilon^2 + 2\Delta} \left\{ \begin{array}{l} \underset{\epsilon\to 0}{\longrightarrow} \frac{\epsilon(1-2\Delta)}{2\Delta} \underset{\Delta\to 0}{\longrightarrow} +0 \text{ (i.e., } \delta_R = 0^\circ) \\ \underset{\Delta\to 0}{\longrightarrow} \frac{1}{\epsilon} \underset{\epsilon\overset{\to}{\pm}0}{\longrightarrow} \pm\infty \text{ (i.e., } \delta_R = \pm 45^\circ) \end{array} \right| .$$

However, the partial-wave amplitudes at $\epsilon = 0$ are, of course, not discontinuous for $x = 1/2$. That is, $\mathrm{Re}A$ and $\mathrm{Im}A$ are the physical quantities. δ and η are parameters in expressions for the physical quantities.

The same general behavior occurs for non-constant widths as shown in Section V. We make the following definition:

Elastic resonance: $x(E = E_r) = 1$

Inelastic resonance:

Absorptive resonance: $1/2 < x(E = E_r) < 1$

Highly absorptive resonance: $0 < x(E = E_r) \leq 1/2$

Other useful relations are:

$$\epsilon = \frac{\mathrm{Im}S_R}{1-\mathrm{Re}S_R} = \frac{\mathrm{Re}A_R}{\mathrm{Im}A_R}$$

$$x = \frac{(\mathrm{Re}A_R)^2+(\mathrm{Im}A_R)^2}{\mathrm{Im}A_R}$$

$$\frac{x}{\epsilon} = \frac{(\mathrm{Re}A_R)^2+(\mathrm{Im}A_R)^2}{\mathrm{Re}A_R} = \frac{2}{\Gamma_{el}(q_{0r}-q_0)}$$

Resonance Width Energy Dependence

All widths are in units of the incident particle mass. We use pion-nucleon scattering in all of the numerical examples.

A. Elastic Width

The simplest assumption would be to set Γ_{el} constant with ϵ. However, the width should have a threshold behavior $k^{2\ell+1}$, so we could use $\Gamma_{el} \propto k^{2\ell+1}$. Resonance theory as given by Breit and Weisskopf^2 gives $\Gamma_{el} = 2\gamma^2 k r_0 V_\ell(k r_0)$, where

γ^2 = reduced width,

r_0 = interaction range (in units of incident particle Compton wavelength), and

$V_\ell(r_0k)$ = barrier penetration factor given by

$$\boxed{V_\ell(r_0k) = \frac{1}{(r_0k)^2\left[j_\ell^2(r_0k)+n_\ell^2(r_0k)\right]} \left\{ \frac{\xrightarrow[k\to 0]{}\left[\frac{r_0^\ell}{1\cdot 3\cdot 5\ldots(2\ell-1)}\right]^2 k^{2\ell}}{\xrightarrow[k\to\infty]{} 1} \right| .} \qquad 16$$

Reference for the spherical Bessel functions $j_\ell(kr_0)$ and $n_\ell(kr_0) = y_\ell(kr_0)$.

For example:

$$\boxed{\begin{gathered} V_0(r_0k) = 1 \ , \ V_1(r_0k) = \frac{(r_0k)^2}{1+(r_0k)^2} \ , \\ V_2(r_0k) = \frac{(r_0k)^4}{9+3(r_0k)^2+(r_0k)^4} \ , \text{ and} \\ V_3(r_0k) = \frac{(r_0k)^6}{225+45(r_0k)^2+6(r_0k)^4+(r_0k)^6} \ . \\ V_4(r_0k) = \frac{(r_0k)^8}{11025+1575(r_0k)^2+135(r_0k)^4+10(r_0k)^6+(r_0k)^8} \end{gathered}}$$

Fig. 3a contains plots of $V_\ell(kr_0)$ for ℓ values from 0 to 5 and $r_0 = 0.71$ ($\simeq 1$ fermi for pion-nucleon scattering). Fig. 3b contains plots for $V_\ell(r_0k)$ for $\ell = 2$ and various values of r_0 .

Layson[4] has derived Γ_{el} by means of the Klein-Gordon equation rather than the Schrödinger equation. His result is

$$\Gamma_{el} = \frac{4M}{q_0 + q_{0_r}} \bar{\gamma}^2 kr_0 V_\ell(r_0k) \ ,$$

where M is the target-particle mass (in units of the incident-particle mass.). Thus, our possible form$\sum$ for Γ_{el} are:

$$\begin{gathered} \text{(a) } \Gamma_{el} = C \\ \text{(b) } \Gamma_{el} = C' k^{2\ell+1} \\ \text{(c) } \Gamma_{el} = 2\gamma^2 kr_0 V_\ell(r_0k) \\ \text{(d) } \Gamma_{el} = \frac{4M}{q_0+q_{0_r}} \bar{\gamma}^2 kr_0 V_\ell(r_0k) \end{gathered} \qquad 17$$

In Fig. 4 we compare the four forms by setting C, C', γ^2 and $\bar{\gamma}^2$ such that $\Gamma_{el}(E = E_r)$ is the same for all four forms. There are two examples. In both examples $\ell = 2, r_0 = 0.71$ and $E_r = 600$ MeV.

(1) $\Gamma_{el}(E = E_r) = 0.5$. (This corresponds to $C = 0.5, C' = 0.00143, \gamma^2 = 0.20702, \bar{\gamma}^2 = 0.10408$.)

(Dashed curves)

(2) $\Gamma_{el}(E = E_r) = 1.0$.(This corresponds to $C = 1.0, C' = 0.00285, \gamma^2 = 0.4404, \bar{\gamma}^2 = 0.20815$.)

(Solid curves)

It is obvious that one cannot use a constant width or $\Gamma_{el} = C' k^{2\ell+1}$ and expect better than 10% accuracy beyond 50 MeV from the resonance position.

In Ref. 1, where the P_{11} partial wave has a zero at about 150 MeV, well below the resonance position at about 560 MeV, a modification of the Layson elastic-width formula is used to allow the zero:

$$\Gamma_{el} = \frac{q_0 - q_z}{q_0} \frac{4M}{q_0 + q_{0_r}} \bar{\gamma}^2 r_0 k V_\ell(r_0 k) \ . \qquad \text{17a}$$

See http://www.roperld.com/science/PionNucleonP11.pdf , where this formula is used.

In Fig. 5a we plot form (d) for Γ_{el} with different values of r_0 . We use $\ell = 2, E_r = 600$ MeV, and $\Gamma_{el}(E = E_r) = 1.0$.

$$r_0 = 0.177 \quad \big(0.25 \text{ fermi}\big) \quad (\bar{\gamma}^2 = 41.66267)$$

$$r_0 = 0.355 \quad \big(0.5 \text{ fermi}\big) \quad (\bar{\gamma}^2 = 1.86624)$$

$$r_0 = 0.71 \quad \big(1.0 \text{ fermi}\big) \quad (\bar{\gamma}^2 = 0.20815)$$

$$r_0 = 1.065 \quad \big(1.5 \text{ fermi}\big) \quad (\bar{\gamma}^2 = 0.09641)$$

$$r_0 = 1.41 \quad \big(2.0 \text{ fermi}\big) \quad (\bar{\gamma}^2 = 0.06439)$$

$$r_0 = 1.775 \quad \big(2.5 \text{ fermi}\big) \quad (\bar{\gamma}^2 = 0.04825)$$

Fig. 5b shows form (d) for $\bar{\gamma}^2 = 0.20815$ and different values of r_0. (This corresponds to $\Gamma_{el}(E = E_r) = 1.0$ for $r_0 = 0.71$.) We use $\ell = 2$ and $E_r = 600$ MeV.

In Fig. 6a we use form (d) and compare Γ_{el} [normalized such that $\Gamma_{el}(E = E_r) = 1.0$] for values of ℓ from 0 to 3. We use $r_0 = 0.71$ and $E_r = 600$ MeV. The values of $\bar{\gamma}^2$ for each ℓ value are:

$$\ell = 0 : \bar{\gamma}^2 = 0.10969 \quad \ell = 2 : \bar{\gamma}^2 = 0.20815$$

$$\ell = 1 : \bar{\gamma}^2 = 0.13058 \quad \ell = 3 : \bar{\gamma}^2 = 0.58443$$

Fig. 6b shows form (d) for values of ℓ from 0 to 3 for $\bar{\gamma}^2 = 0.10969$. (This corresponds to $\Gamma_{el}(E = E_r) = 1.0$ for $\ell = 0$.) We use $r_0 = 0.71$ and $E_r = 600$ MeV.

B. Inelastic Width

Below threshold $(k = k_0)$ for inelastic scattering, $\Gamma_{in} = 0$. (For pion-nucleon scattering $k_0 = 1.479$.) So the simplest assumption would be $\Gamma_{in} = C_{in}\theta(k - k_0)$, where we use the step function

$$\theta(k - k_0) = \left\{ \begin{array}{l} 1 \text{ for } k \geq k_0 \\ 0 \text{ for } k < k_0 \end{array} \right| \ .$$

Or, we could give it the threshold behavior $\Gamma_{in} = C' \ \theta(k - k_0)(k - k_0)^{2\ell+1}$ .. The problem here is that we do not know what value of ℓ to use. Assume that the inelastic final state is a two-body state. This final state may have a different orbital angular momentum (ℓ') than the initial state; the final state value is the one that should be used. For example, the process

$\pi + p \rightarrow \sigma + p$, where σ is a 0^{++} low-mass meson,

may be the dominant mechanism for inelastic scattering in the P_{11} pion-nucleon state. If so, the initial $\ell = 1$ and the final $\ell' = 0$. Thus the threshold behavior should be $(k - k_0)$ rather than $(k - k_0)^3$. In the numerical examples given below we assume that the final and initial $\ell's$ are identical.

Analogous to the elastic case, we could use the following forms for Γ_{in} :

$$\text{(a) } \Gamma_{in} = C_{in}\theta(k-k_0)$$
$$\text{(b) } \Gamma_{in} = C'_{in}(k-k_0)^{2\ell+1}\theta(k-k_0)$$
$$\text{(c) } \Gamma_{in} = 2\gamma_{in}^2(k-k_0)r_0V[(k-k_0)r_0]\theta(k-k_0)$$
$$\text{(d) } \Gamma_{in} = \tfrac{4M}{q_0+q_{0r}}\bar{\gamma}_{in}^2(k-k_0)r_0V[(k-k_0)r_0]\theta(k-k_0)$$

Actually, for a two-body inelastic final state one should use k' , the final-state c.m. momentum, rather than $(k-k_0)$ .Its relation to the total c.m. energy W is

$$k' = \frac{1}{2W}\sqrt{\left[W^2+\left(\frac{M'_1}{M_1}\right)^2-\left(\frac{M'_2}{M_1}\right)^2\right]^2-4\left(\frac{M'_1}{M_1}\right)^2W^2}$$

(in units of incident-particle mass), where the final state particle masses are M'_1 and M'_2 (in units of the incident-particle mass). In terms of these final-state masses, the inelastic threshold c.m. momentum is

$$k = M\sqrt{\frac{E(E+2)}{(M+1)^2+2ME}}\ ,$$

where the threshold incident particle lab. kinetic energy (in units of the incident-particle mass) is

$$E = \frac{(M'_1+M'_2)^2-(M+1)^2}{2M}\ .$$

$$\therefore \boxed{k = \tfrac{1}{2(M'_1+M'_2)}\sqrt{\left[(M'_1+M'_2)^2-(M+1)^2\right]\left[(M'_1+M'_2)^2-(M+1)^2+4M\right]}}$$

When there are several possible two-body inelastic final states[2],

$$\Gamma_{in} = \sum_i \Gamma_{in}(M'_{1i}, M'_{2i})$$

where the sum goes over all possible final states. Of course, there may be three- or many-body inelastic final states possible, also. For purposes of illustration we use $(k-k_0)$ rather than k' and only one inelastic width.

In Fig. 7 we compare the four forms for the energy dependence of Γ_{in} . We set $\Gamma_{in}(E=E_r) = 0.5$ for all four forms, which dictates that $C_{in} = 0.5, C'_{in} = 0.03059, \gamma_{in}^2 = 1.35616,$ and $\bar{\gamma}_{in}^2 = 0.68179$. We use $\ell = 2,\ r_0 = 0.71$ $k_0 = 1.479$ and $E_r = 600$ MeV.

In Fig. 8a we plot form (d) for Γ_{in} with different values of r_0 . We use $\ell = 2, E_r = 600$ MeV and $\Gamma_{in}(E=E_r) = 0.5$.

$r_0 = 0.177$ $\left(0.25 \text{ fermi}\right)$ $(\overline{\gamma}_{in}^2 = 411.41793)$

$r_0 = 0.355$ $\left(0.5 \text{ fermi}\right)$ $(\overline{\gamma}_{in}^2 = 14.05116)$

$r_0 = 0.71$ $\left(1.0 \text{ fermi}\right)$ $(\overline{\gamma}_{in}^2 = 0.68179)$

$r_0 = 1.065$ $\left(1.5 \text{ fermi}\right)$ $(\overline{\gamma}_{in}^2 = 0.17638)$

$r_0 = 1.41$ $\left(2.0 \text{ fermi}\right)$ $(\overline{\gamma}_{in}^2 = 0.08855)$

$r_0 = 1.775$ $\left(2.5 \text{ fermi}\right)$ $(\overline{\gamma}_{in}^2 = 0.05703)$

Fig. 8b shows form (d) for different values of r_0 and $\overline{\gamma}_{in}^2 = 0.68179$. (This corresponds to $\Gamma_{in}(E = E_r) = 0.5$ for $r_0 = 0.71$.) We use $\ell = 2$ and $E_r = 600$ MeV.

In Fig. 9a we use form (d) and compare Γ_{in} , [normalized such that $\Gamma_{in}(E = E_r) = 0.5$] for values of ℓ from 0 to 3. We use $r_0 = 0.71, k_0 = 1.479$ and $E_r = 600$ MeV. The values of $\overline{\gamma}_{in}^2$ for each ℓ value are:

$\ell = 0 : \overline{\gamma}_{in}^2 = 0.10123$ $\ell = 2 : \overline{\gamma}_{in}^2 = 0.68179$

$\ell = 1 : \overline{\gamma}_{in}^2 = 0.16691$ $\ell = 3 : \overline{\gamma}_{in}^2 = 8.63331$

Fig. 9b shows form (d) for values of ℓ from 0 to 3. for $\overline{\gamma}_{in}^2 = 0.10123$. (This corresponds to $\Gamma_{in}(E = E_r) = 0.5$ for $\ell = 0$.) We use $r_0 = 0.71, k_0 = 1.479$ and $E_r = 600$ MeV.

C. Important Observation

Figs. 4 and 7 make it evident that the choice of the resonance-width energy dependence may be important, even in the vicinity of the resonance position.

V. Numerical Examples of Resonance Phase Shifts, Absorption Parameters, and Partial-Wave Amplitudes

A. Comparison of the Four Forms for the Width Energy Dependence

Four numerical examples are shown in Fig. 10 in each of which we compare the four forms for the energy dependence of the elastic and inelastic widths. In all four examples $\ell = 2, \Gamma(E = E_r) = 1.0, r_0 = 0.71$ $k_0 = 1.479$ and $E_r = 600$ MeV. The first two examples are in Fig. 10a and the last two are in Fig. 10b.

(1) $\Gamma_{el}(E = E_r) = 1.0, \Gamma_{in}(E = E_r) = 0$.(This corresponds to $C = 1.0, C' = 0.00285, \gamma^2 = 0.41404, \overline{\gamma}^2 = 0.20815$ and $C_{in} = C_{in}' = \gamma_{in}^2 = \overline{\gamma}_{in}^2 = 0$.) Of course, $\eta = 1$ everywhere for this example and $\mathrm{Re}A$ and $\mathrm{Im}A$ reach their unitary limits. (Solid curves)

(2) $\Gamma_{el}(E = E_r) = 0.5, \Gamma_{in}(E = E_r) = 0.5$.(This corresponds to $C = 0.5, C' = 0.3, \gamma^2 = 0.20702, \overline{\gamma}^2 = 0.10408$ and $C_{in} = 0.5, C_{in}' = 0.03059, \gamma_{in}^2 = 1.35616, \overline{\gamma}_{in}^2 = 0.68179$.) The discontinuity in δ at $\epsilon = 0, x = 1/2$ is plainly visible. Note that the peak of $\mathrm{Im}A$ is shifted to the low-energy side in the case of non-constant widths. (Dashed curves)

(3) $\Gamma_{el}(E = E_r) = 0.25, \Gamma_{in}(E = E_r) = 0.75$.(This corresponds to $C = 0.25, C' = 0.00071, \gamma^2 = 0.10351, \overline{\gamma}^2 = 0.05204$ and $C_{in} = 0.75, C_{in}' = 0.04588, \gamma_{in}^2 = 2.03423, \overline{\gamma}_{in}^2 = 1.02268$.) Form (a) for this example and the

next example give identical $\eta's$. Note that the dip in η is shifted to the low-energy side in the cases of non-constant widths. (Dashed curves)

(4) $\Gamma_{el}(E=E_r)=0.75, \Gamma_{in}(E=E_r)=0.25$.(This corresponds to $C=0.75, C'=0.00214, \gamma^2=0.31053, \overline{\gamma}^2=0.15611$ and $C_{in}=0.25, C'_{in}=0.01529, \gamma^2_{in}=0.67808, \overline{\gamma}^2_{in}=0.34089$.) Form (a) for this example and the previous example give identical $\eta's$. Note that the dip in η is shifted to the low-energy side in the cases of non-constant widths. (Solid curves)

(B) Comparison for Various Values of x Using Form (d) for the Width Energy Dependence

From here on all examples will use form (d) for the elastic and inelastic widths. Nine numerical examples are shown in Fig. 11a for nine different values of $x=\Gamma_{el}/\Gamma$ at $E=E_r$. In all nine examples $\ell=2, \Gamma(E=E_r)=1.0, r_0=0.71, k_0=1.479$ and $E_r=600$ MeV.

(1) $x(E=E_r)=0.0$. (This corresponds to $\overline{\gamma}^2=0$ and $\overline{\gamma}^2_{in}=$ arbitrary) This example is no elastic scattering at all ($A=0, \delta=0, \eta=1$).

(2) $x(E=E_r)=0.125$. (This corresponds to $\overline{\gamma}^2=0.02602$ and $\overline{\gamma}^2_{in}=1.19303$).

(3) $x(E=E_r)=0.25$. (This corresponds to $\overline{\gamma}^2=0.05204$ and $\overline{\gamma}^2_{in}=1.022668$).

(4) $x(E=E_r)=0.375$. (This corresponds to $\overline{\gamma}^2=0.07806$ and $\overline{\gamma}^2_{in}=0.85223$).

(5) $x(E=E_r)=0.5$. (This corresponds to $\overline{\gamma}^2=0.10408$ and $\overline{\gamma}^2_{in}=0.68179$). The discontinuity at $\epsilon=0, x=1/2$ is plainly visible.

(6) $x(E=E_r)=0.625$. (This corresponds to $\overline{\gamma}^2=0.13010$ and $\overline{\gamma}^2_{in}=0.51134$).

(7) $x(E=E_r)=0.75$. (This corresponds to $\overline{\gamma}^2=0.15611$ and $\overline{\gamma}^2_{in}=0.34089$).

(8) $x(E=E_r)=1.0$. (This corresponds to $\overline{\gamma}^2=0.20815$ and $\overline{\gamma}^2_{in}=0$). This example is pure elastic scattering ($\eta=1$).

In Fig. 11b we compare elastic resonances $[x(E=E_r)=1]$ for different values of $\overline{\gamma}^2$. We use $\ell=2, r_0=0.71,$ and $E_r=600$ MeV.

(1) $\Gamma(E=E_r)=1.5\ (\overline{\gamma}^2=0.31223)$

(2) $\Gamma(E=E_r)=1.25\ (\overline{\gamma}^2=0.26019)$

(3) $\Gamma(E=E_r)=1.0\ (\overline{\gamma}^2=0.20815)$

(4) $\Gamma(E=E_r)=0.75\ (\overline{\gamma}^2=0.15611)$

(5) $\Gamma(E=E_r)=0.5\ (\overline{\gamma}^2=0.10408)$

(2) $\Gamma(E=E_r)=0.25\ (\overline{\gamma}^2=005204)$

Comparison for Various Values of r_0

In the examples given here $\ell=2, k_0=1.479,$ and $E_r=600$ MeV. We use form (d) for widths. Numerical examples for different values of r_0 at two values of $x(E=E_r)$ are given in Fig. 12a.

(1) $\Gamma_{el}(E=E_r)=0.5$ and $\Gamma_{in}(E=E_r)=1.0$. That is:

$\overline{\gamma}^2=0.93312$ and $\overline{\gamma}^2_{in}=28.10232$ for $r_0=0.355$ $\left(0.5 \text{ fermi}\right)$
$\overline{\gamma}^2=0.10408$ and $\overline{\gamma}^2_{in}=1.36357$ for $r_0=0.71$ $\left(1.0 \text{ fermi}\right)$
$\overline{\gamma}^2=0.04820$ and $\overline{\gamma}^2_{in}=0.35275$ for $r_0=1.065$ $\left(1.5 \text{ fermi}\right)$
$\overline{\gamma}^2=0.03220$ and $\overline{\gamma}^2_{in}=0.17710$ for $r_0=1.41$ $\left(2.0 \text{ fermi}\right)$

(2) $\Gamma_{el}(E = E_r) = 1.0$ and $\Gamma_{in}(E = E_r) = 0.5$. That is:

$\overline{\gamma}^2 = 1.86624$ and $\overline{\gamma}_{in}^2 = 14.05116$ for $r_0 = 0.355$ $\big(0.5$ fermi$\big)$
$\overline{\gamma}^2 = 0.20815$ and $\overline{\gamma}_{in}^2 = 0.68179$ for $r_0 = 0.71$ $\big(1.0$ fermi$\big)$
$\overline{\gamma}^2 = 0.09641$ and $\overline{\gamma}_{in}^2 = 0.17638$ for $r_0 = 1.065$ $\big(1.5$ fermi$\big)$
$\overline{\gamma}^2 = 0.06439$ and $\overline{\gamma}_{in}^2 = 0.08855$ for $r_0 = 1.41$ $\big(2.0$ fermi$\big)$

Examples for different values of r_0 with $\overline{\gamma}^2 = 0.20815$ and $\overline{\gamma}_{in}^2 = 0.68179$ are given in Fig. 12b. (This corresponds to $\Gamma_{el}(E = E_r) = 1.0$ and $\Gamma_{in}(E = E_r) = 0.5$ for $r_0 = 0.71$.)

D. Comparison for Various Values of k_0

In the examples given here $\ell = 2, r_0 = 0.71,$ and $E_r = 600$ MeV. We use form (d) for the widths. Numerical examples for different values of k_0 at two values of $x(E = E_r)$ are given in Fig. 13a.

(1) $\Gamma_{el}(E = E_r) = 0.5$ and $\Gamma_{in}(E = E_r) = 1.0$. That is:

$\overline{\gamma}^2 = 0.10408$ and $\overline{\gamma}_{in}^2 = 1.36357$ for $k_0 = 1.479$ $\big(161$ MeV = one-pion production$\big)$
$\overline{\gamma}^2 = 0.10408$ and $\overline{\gamma}_{in}^2 = 22.9684$ for $k_0 = 2.315$ $\big(344$ MeV = two-pion production$\big)$
$\overline{\gamma}^2 = 0.10408$ and $\overline{\gamma}_{in}^2 = 70.65652$ for $k_0 = 2.507$ $\big(393$ MeV = N_{33}^* production$\big)$

(1) $\Gamma_{el}(E = E_r) = 1.0$ and $\Gamma_{in}(E = E_r) = 0.5$. That is:

$\overline{\gamma}^2 = 0.20815$ and $\overline{\gamma}_{in}^2 = 0.68179$ for $k_0 = 1.479$ $\big(161$ MeV = one-pion production$\big)$
$\overline{\gamma}^2 = 0.20815$ and $\overline{\gamma}_{in}^2 = 11.48477$ for $k_0 = 2.315$ $\big(344$ MeV = two-pion production$\big)$
$\overline{\gamma}^2 = 0.20815$ and $\overline{\gamma}_{in}^2 = 35.32826$ for $k_0 = 2.507$ $\big(393$ MeV = N_{33}^* production$\big)$

Examples for different values of k_0 with $\overline{\gamma}^2 = 0.20815$ and $\overline{\gamma}_{in}^2 = 0.68179$ are given in Fig. 13b. (This corresponds to $\Gamma_{el}(E = E_r) = 1.0$ and $\Gamma_{in}(E = E_r) = 0.5$ for $k_0 = 1.479$.)

E. Comparison for Various Values of E_r

In the examples given here $\ell = 2, r_0 = 0.71,$ and $k_0 = 1.479$. We use form (d) for the widths. Numerical examples for different values of E_r at two values of $x(E = E_r)$ are given in Fig. 14a.

(1) $\Gamma_{el}(E = E_r) = 0.5$ and $\Gamma_{in}(E = E_r) = 1.0$. That is:

$\overline{\gamma}^2 = 0.15699$ and $\overline{\gamma}_{in}^2 = 9.47628$ for $E_r = 400$ MeV
$\overline{\gamma}^2 = 0.10408$ and $\overline{\gamma}_{in}^2 = 1.36357$ for $E_r = 600$ MeV
$\overline{\gamma}^2 = 0.08509$ and $\overline{\gamma}_{in}^2 = 0.57269$ for $E_r = 800$ MeV

(2) $\Gamma_{el}(E = E_r) = 1.0$ and $\Gamma_{in}(E = E_r) = 0.5$. That is:

$\overline{\gamma}^2 = 0.31397$ and $\overline{\gamma}_{in}^2 = 4.73814$ for $E_r = 400$ MeV
$\overline{\gamma}^2 = 0.20815$ and $\overline{\gamma}_{in}^2 = 0.68179$ for $E_r = 600$ MeV
$\overline{\gamma}^2 = 0.17018$ and $\overline{\gamma}_{in}^2 = 0.28634$ for $E_r = 800$ MeV

Examples for different values of E_r with $\overline{\gamma}^2 = 0.20815$ and $\overline{\gamma}_{in}^2 = 0.68179$ are given in Fig. 14b. (This corresponds to $\Gamma_{el}(E = E_r) = 1.0$ and $\Gamma_{in}(E = E_r) = 0.5$ for $E_r = 600$ MeV.)

F. Comparison for various values of ℓ

In the examples that follow $r_0 = 0.71, k_0 = 1.479$ and $E_r = 600$ MeV. We use form (d) for the widths. Examples for different values of ℓ at two values of $x(E = E_r)$ are given in Fig. 15a.

(1) $\Gamma_{el}(E = E_r) = 0.5$ and $\Gamma_{in}(E = E_r) = 1.0$. That is:

$\bar{\gamma}^2 = 0.05485$ and $\bar{\gamma}^2_{in} = 0.20247$ for $\ell = 0$
$\bar{\gamma}^2 = 0.06529$ and $\bar{\gamma}^2_{in} = 0.33383$ for $\ell = 1$
$\bar{\gamma}^2 = 0.10408$ and $\bar{\gamma}^2_{in} = 1.36357$ for $\ell = 2$
$\bar{\gamma}^2 = 0.29222$ and $\bar{\gamma}^2_{in} = 17.26663$ for $\ell = 3$

(1) $\Gamma_{el}(E = E_r) = 1.0$ and $\Gamma_{in}(E = E_r) = 0.5$. That is:

$\bar{\gamma}^2 = 0.10969$ and $\bar{\gamma}^2_{in} = 0.10123$ for $\ell = 0$
$\bar{\gamma}^2 = 0.13058$ and $\bar{\gamma}^2_{in} = 0.16691$ for $\ell = 1$
$\bar{\gamma}^2 = 0.20815$ and $\bar{\gamma}^2_{in} = 0.68179$ for $\ell = 2$
$\bar{\gamma}^2 = 0.58443$ and $\bar{\gamma}^2_{in} = 8.6331$ for $\ell = 3$

Examples for different values of ℓ with $\bar{\gamma}^2 = 0.10969$ and $\bar{\gamma}^2_{in} = 0.10123$ are given in Fig. 15b. (This corresponds to $\Gamma_{el}(E = E_r) = 1.0$ and $\Gamma_{in}(E = E_r) = 0.5$ for $\ell = 0$.)

G. Important Observations

In Fig. 11 we see that, even though the phase shift may be small for an inelastic resonance, there is distinctive behavior that may enable one to identify such a resonance in a phase-shift analysis; viz., the phase shift passes downward through zero at the resonance position and the absorption parameter has a deep dip slightly to the low-energy side of the resonance position. Stated in terms of the partial-wave amplitude the same words apply for an elastic resonance; viz., the real part of the amplitude passes downward through zero at the resonance position and the imaginary part has a peak slightly to the low-energy side of the resonance position. Of course, a large background in the same state as the resonance could obliterate some of these distinguishing features.

More Kinematics

Here are some more useful equations:

$$\boxed{s = W^2 = \left(\sqrt{k^2+m^2} + \sqrt{k^2+M^2}\right)^2 = (q_0+p_0)^2 = (m+M)^2 + 2ME}$$

$$\boxed{k = \sqrt{\frac{\left(s-m^2-M^2\right)^2-4m^2M^2}{4s}} = \frac{1}{2W}\sqrt{W^4-2(M^2+m^2)W^2+\left(M^2-m^2\right)^2} = \frac{1}{2W}\sqrt{\left[W^2-(M+m)^2\right]\left[W^2-(M-m)^2\right]}}$$

$$\boxed{q_0 = \frac{1}{2W}\sqrt{W^4-2(M^2-m^2)W^2+\left(M^2-m^2\right)^2} = \frac{1}{2W}\left[W^2-(M^2-m^2)\right]}$$

$$\therefore \boxed{p_0 = W - q_0 = \frac{1}{2W}\left[W^2+(M^2-m^2)\right]}$$

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